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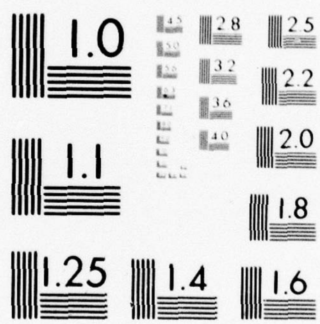

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ASYMPTOTIC ANALYSIS OF

OFF-CENTER UNSTABLE

CONFOCAL RESONATORS

THESIS

AFIT/GEP/PH/79D-4 Michael P. Grone
Captain USAF

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ASYMPTOTIC ANALYSIS OF OFF-CENTER UNSTABLE CONFOCAL RESONATORS

9 Master's THESIS

Presented to the Faculty of the School of Engineering

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Air University

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Requirements for the Degree of

Master of Science

by

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Graduate Engineering Physics

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Preface

During my study of the asymptotic analysis of unstable resonators, I have been helped and encouraged by many people. I am indebted to my advisor, Major Glenn Doughty, for the effort, concern, and long hours he has spent with me. I would also like to thank the other members of my thesis committee, Major John Erkkila and Doctor Donn Shankland, for their helpful comments and advice. I want to extend my appreciation to Captain Russ Sorrenson and Lieutenant Mark Franz whose suggestions and helpful conversation smoothed the rough spots many times. Finally, I wish to thank my family. Their confidence and concern are always a source of strength for me.

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a	distance of lower feedback mirror edge from optical axis.
a_n	x coordinate for nth virtual image of the (d,-a) feedback mirror edge.
α	angle to optical axis of ray from edge (d,-a) to point on large mirror.
α_{an}	angle to optical axis of ray from edge virtual image point $(\rho_{-n},-a_{-n})$ to edge (d,-a)
α_{bn}	angle to optical axis of ray from edge virtual image point (ρ_{-n},b_{-n}) to edge point (d,-a)
b	distance of upper feedback mirror edge from optical axis.
b_n	x coordinate for nth virtual image of the (d,b) feedback mirror edge.
β	angle to optical axis of ray from edge (d,b) to point on large mirror.
β_{an}	angle to optical axis of ray from edge virtual image point $(\rho_{-n},-a_{-n})$ to edge (d,b).
β_{bn}	angle to optical axis of ray from edge virtual image point (ρ_{-n},b_{-n}) to edge (d,b).
c	velocity of light in a vacuum.
C_1	feedback mirror center of curvature.
C_2	large mirror center of curvature

d	distance between feedback mirror and focal point.
D	separation distance between mirrors
E	total field in resonator
f.p.	focal point
$f(z,x)$	modulated amplitude function of right-traveling wave
$\hat{f}(\rho,x)$	geometrical amplitude function of right-traveling wave.
F_{ea}	lower effective Fresnel number of off-center feedback mirror.
F_{eb}	upper effective Fresnel number of off-center feedback mirror.
F_{eff}, F_e	effective Fresnel number of centered feedback mirror.
$g(\rho,\theta)$	modulated amplitude function of left-traveling wave.
$\hat{g}(\rho,x)$	geometrical amplitude function of left-traveling wave.
G	gain coefficient of medium
k	wave number
$k_{an}(x)$	x-coordinate of r_1 amplitude function that will propagate to location x on feedback mirror on nth round trip of resonator.
$k_{bn}(x)$	x-coordinate of s_1 amplitude function that will propagate to location x on feedback mirror on nth round trip of resonator.
M	geometrical magnification of resonator.
M_1	feedback mirror.
M_2	large mirror.

μ	permeability of medium, or eigenvalue.
$q(x)$	auxiliary function of diffraction amplitude in centered resonator case.
$q_a(x)$	auxiliary function of diffraction amplitude function s_1 in off-centered resonator case.
$q_b(x)$	auxiliary function of diffraction amplitude function r_1 in off-centered resonator case.
Q	constant gain coefficient for one traverse across resonator.
$r_n(\rho,x)$	diffraction amplitude function of right-traveling wave due to edge (d,b).
R_1	radius of curvature of feedback mirror.
R_2	radius of curvature of large mirror.
ρ	cylindrical radial coordinate from focal point.
ρ_n	ρ-coordinate of virtual image of feedback mirror edge.
$s_n(\rho,x)$	diffraction amplitude function of right-traveling wave due to edge (d,-a).
σ	conductivity of medium.
θ	cylindrical angular coordinate from optical axis.
$u_n(\rho,x)$	diffraction amplitude function of left-traveling wave due to edge (d,b).
$v_n(\rho,x)$	diffraction amplitude function of left-traveling wave due to edge (d,-a).
x	cartesian coordinate orthogonal to optical axis.
z	cartesian coordinate along optical axis.

Abstract

A polynomial equation for the eigenvalues of the modes of off-center unstable confocal resonators is developed. A constant gain for steady state modes in a bare cavity is assumed. The field is built-up from right and left-traveling diffraction components for a number of round trips through the resonator and geometrical components from the core region. Using an asymptotic expansion of the diffraction integral, the boundary conditions are developed. These, with the propagation equations across the resonator, are used to relate the diffraction and geometrical components to the diffraction amplitude after one round trip in the cavity. The polynomial equation for the eigenvalues is developed from the first round trip amplitude function, after approximating a slowly varying function of the field to be constant. A method is proposed for examining the behavior of the approximated function for the centered resonator case and including it in mode calculations if necessary.

ASYMPTOTIC ANALYSIS OF OFF-CENTER UNSTABLE CONFOCAL RESONATORS

I. Introduction

Unstable resonators have many attractive features including large mode volume, substantial transverse mode discrimination, and direct output coupling (Ref 9:156). Due to these characteristics, the determination of the resonant modes of these resonators has been the subject of much research. Unlike stable resonator modes (Ref 10:328-332), those of the unstable resonator cannot be easily described in terms of well-known special functions.

Many methods have been used to determine the modes of unstable configurations. Geometric approaches (Ref 9:156) yield crude results for only the lowest-loss mode. These results are valid only for high Fresnel number resonators. Methods which iterate to self-consistent field distributions (Ref 2) typically yield only lowest-loss mode information. More elaborate techniques have been developed which employ Fast-Fourier-Transform techniques and include effects of gain, diffraction, and index of refraction variations (Ref 13). This technique, however, is not practical for high Fresnel number resonators due to prohibitive increases in the required computing time. Other techniques using numerical

integration also become costly to run for high Fresnel number resonators.

An asymptotic theory of unstable resonator modes using an asymptotic expansion in the resonator integral equation has been deceloped by Horwitz (Ref 3) which yields the lowest-loss mode and higher order modes, uses little computer time, and is valid for a wide range of Fresnel numbers. The eigenvalues of the modes are found as the roots of a polynomial equation in this method. Horwitz has extended this theory to include misalignment of the feedback mirror by a small tilt (Ref 4).

Moore and McCarthy (Ref 6) have extended the asymptotic approach of Horwitz to determine the steady state modes of a bare resonator and a loaded resonator with arbitrary fixed gain distributions. Their analysis applies to the positive branch unstable strip confocal resonator (Ref 9:355). For this geometry the optical axis passes through the center of both mirrors. This centered confocal resonator geometry is chosen because the beam is not focused within the cavity, also the output beam should be well collimated since in the geometric limit it would be a plane wave.

In comparison to the centered resonator geometry, resonators with an off-center feedback mirror appear to offer advantage in beam quality. Better mode-loss separation and more far field energy in the first Airy square (for rectangular mirror cavities) may be achieved (Refs 7:2159 and 14:1828). To examine these characteristics, a method of

determining the modes for off-center unstable confocal resonators is needed.

The primary objective of this work is to extend the analysis of Moore and McCarthy to obtain a polynomial equation for the eigenvalues of the off-center geometry. A secondary objective is to examine the approximations made by Moore and McCarthy in their analysis of the centered geometry and to report the results of this work.

This report consists of six chapters and five appendices. Following this introduction, Chapter II presents the basic formalism, assumptions, and general approach used in this analysis. Chapter III develops the equations for the boundary conditions on the mirrors while the amplitude functions for the field in the resonator are developed in Chapter IV. The polynomial equation for the eigenvalues of the off-center unstable resonator modes is developed in Chapter V. An approximation made by Moore and McCarthy to obtain the polynomial equation for the centered case is also examined in this chapter. The last chapter presents specific conclusions and recommendations. The appendices include mathematical derivations and detailed discussions supporting the work presented in the main body of this report.

II. Problem Formulation

The purpose of this chapter is to set the framework for use in the following development. Four major areas are discussed. The first area is the basic formalism for the problem. This includes a discussion of the geometry, assumptions, and general approach used in the development. The second section presents the basic form assumed for the field in the cavity. Next, gain in the cavity is considered in light of assumptions made for steady state modes to be supported. Finally, the assumed form of the field amplitude expressions is discussed.

Basic Formalism

Geometry. The geometry of the off-center unstable strip confocal resonator considered in this report is illustrated in Fig. (1). This cavity consists of a convex cylindrical feedback mirror at the right end and a large concave cylindrical mirror at the left end. The feedback mirror, M_1, is located distance d to the left of the common focal point, f.p. The large mirror is displaced a distance D to the left of the feedback mirror along the optical axis. The optical axis is taken as the line joining the centers of curvature of the mirrors, C_1 and C_2 .

The upper edge of the feedback mirror is located distance b from the optical axis and the lower edge distance

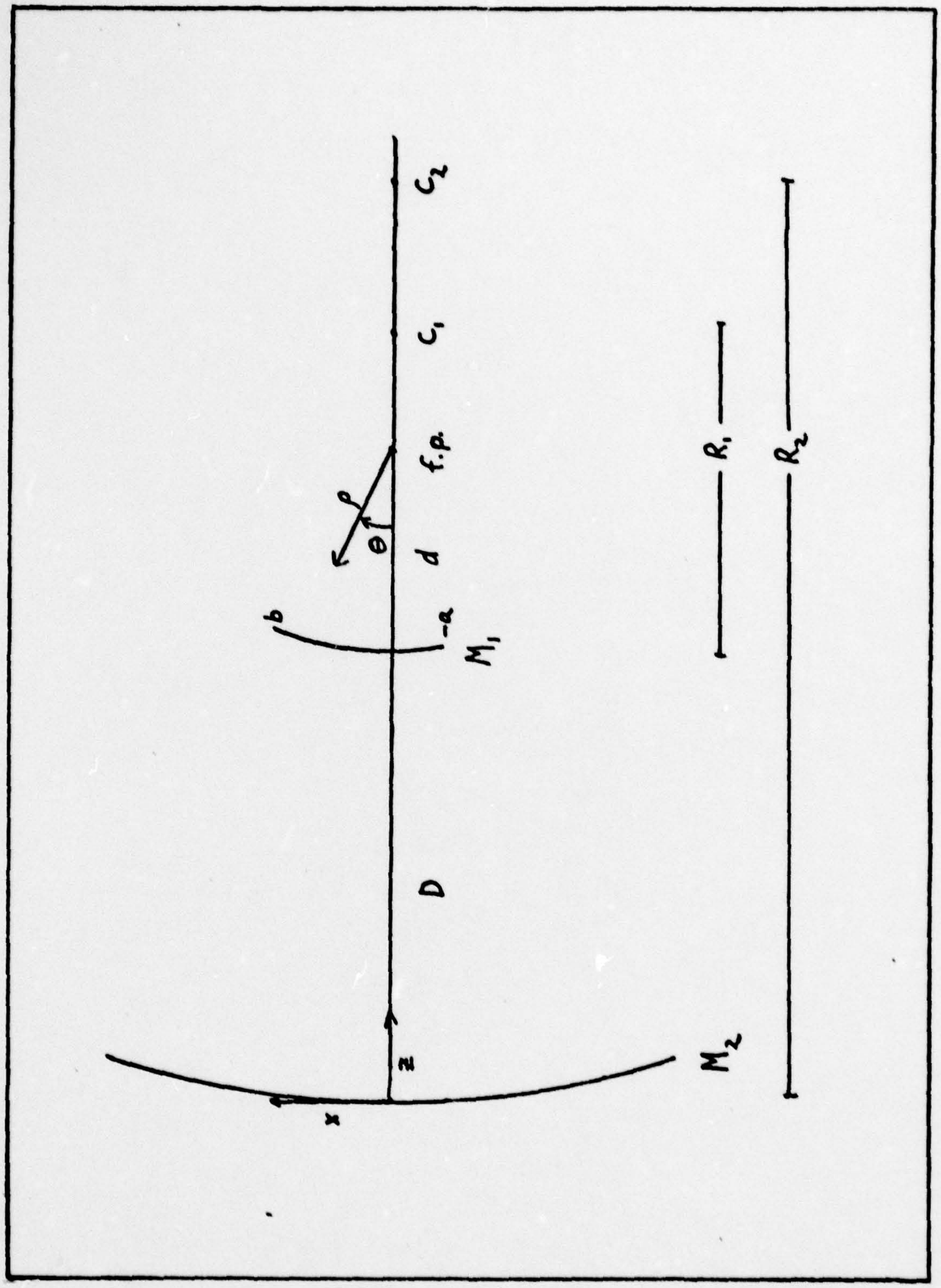

Fig 1 OFF-CENTER UNSTABLE CONFOCAL GEOMETRY

-a. For this development it is assumed that $|b|>|a|$. The transverse dimensions of the large mirror are assumed to be sufficiently large so that all of the incident radiation is reflected without introduction of diffraction effects. Practically, this "sufficiently large" transverse dimension can be considered to be at least $b(M+1)$ (Ref 6:228) where M is the resonator magnification developed in Appendix B and given by

$$M = \frac{D+d}{d} \tag{1}$$

Two effective Fresnel numbers can be defined for this cavity due to the asymmetry of the feedback mirror edges to the optical axis. They are

$$F_{eb} = \frac{kb^2}{4\pi d} \tag{2}$$

$$F_{ea} = \frac{ka^2}{4\pi d} \tag{3}$$

These are also reffered to as equivalent Fresnel numbers (Refs 3:1529 and 12:39).

The asymmetry caused by the feedback mirror not being centered on the optical axis can be introduced by starting with the centered geometry and rotating the feedback mirror about its center of curvature. The difference between the asymmetry being introduced by a rotation rather than a tilting of the feedback mirror is discussed in Appendix A. It

is clear that when the feedback mirror edge distances from the optical axis are equal, $|b|=|a|$, the expressions developed in this analysis should reduce to those of the centered case.

As in Moore and McCarthy's analysis (Ref 6:228), the field in the cavity can be modeled by a right-traveling modulated plane wave and a left-traveling modulated cylindrical wave originating at the focal point. To describe these fields conveniently, two coordinate systems are used, see Fig. (1). A cartesian coordinate system (z,x) with its origin at the intersection of the optical axis with the large mirror, has positive z defined to the right of the mirror and positive x above the axis. The cylindrical coordinate system (ρ,θ) has its origin at the focal point. Positive θ is defined as a clockwise rotation in the (z,x) plane about the focal point with $\theta=0$ along the optical axis to the left of the focal point.

When the paraxial approximation is valid these coordinate systems are related by

$$\rho = Md-z \tag{4}$$

$$\theta = \frac{x}{\rho} \tag{5}$$

Then the paraxial coordinates (ρ,x) can be used to describe locations relative to the focal point.

<u>Assumptions</u>. To make the development more tractable,

without sacrificing accuracy, five major assumptions are made:

1. The field is scaler.
2. The mirrors are perfectly conducting.
3. The laser is operating in steady state.
4. The main contributions to the modes are from paraxial regions in the resonator.
5. The optical axis does not pass closer than several Fresnel zones from the feedback mirror edge.

The first assumption greatly simplifies the analysis of the problem. The scalar equations neglect the vector nature of the field and so polarization effects are not modeled. This allows the use of the strip resonator geometry. In a bare cavity analysis this is not a serious drawback. If a specific gain medium was being modeled then polarization becomes a factor.

The assumption of perfectly conducting mirrors implies that all of the incident light is reflected. Thus losses due to absorption by the mirrors are not modeled.

Oscillation in steady state at a single frequency neglects time-dependent phenomena. It also implies a constant gain in the resonator that exactly cancels losses due to output coupling. This analysis differs from that of a truly bare cavity (no gain medium) which would exhibit decaying modes.

The assumption of paraxial modes implies that the major contribution to the mode comes from a paraxial region about the geometric source point.

It is assumed that the optical axis does not pass closer than several Fresnel zones from the edge of the feedback mirror because of the asymptotic expansion used in this analysis. This expansion is carried out to terms of the order of the inverse Fresnel number for the edge nearest the optical axis. If the optical axis passes closer than several Fresnel zones this expansion becomes increasingly inaccurate.

Approach. The general approach to obtaining the polynomial equation for the mode eigenvalues consists of four parts. The first part assumes a form for the field in the cavity built up from a geometric contribution and diffraction components due to the feedback mirror edges. The boundary conditions relating these field compenents are developed in the second part of the analysis. In this section the development of the boundary conditions on the feedback mirror involves the use of an asymptotic approximation for a diffraction integral evaluated over the limits of the mirror. The third part of the analysis develops expressions relating the geometrical components and diffraction components of the amplitude function to the diffraction components after one round trip in the resonator. The diffraction amplitude components are then expressed in terms of an auxiliary slowly varying function. In the fourth part an approximation of the auxiliary function, similar to Moore and McCarthy's (Ref6:233), is made to obtain a polynomial for the eigenvalues. This section then returns to the centered case and proposes a matrix

equation that could be used to examine the auxiliary function approximation and determine its validity.

Field Considerations

Field in Resonator. The field inside the cavity is assumed to consist of a right-traveling wave and left-traveling wave similar in form to those of geometrical theories (Ref 9: 157). The right-traveling wave has the form of a modulated plane wave where its amplitude function includes rapidly oscillating transverse phase terms due to diffraction effects. Similarly, the left-traveling wave assumes the form of a modulated cylindrical wave originating at the focal point of the cavity. Its amplitude function also includes diffraction effects. The total field in the cavity is then given by

$$E = \left[f(z,x)\exp\{ik(z+Md)\} - \rho^{-\frac{1}{2}} g(\rho,\theta)\exp\{ik\rho\} \right] e^{-iwt} \tag{6}$$

The mirrors of the cavity impose boundary conditions upon the field. At the mirrors the phases of the right and left-traveling waves must match. Also, the total field must vanish on the two mirrors. This yields the boundary conditions

$$f(0,M\theta d) = (Md)^{-\frac{1}{2}} g(Md,\theta) \tag{7}$$

$$g(d,\theta) = \begin{cases} d^{\frac{1}{2}} f(D,\theta d)e^{2ikD} & ;-\frac{a}{d}<\theta<\frac{b}{d} \\ 0 & ;\text{otherwise} \end{cases} \tag{8}$$

Gain Considerations

From the assumptions used in this analysis, the field must obey the scalar Helmholtz equation

$$\nabla^2 E = \frac{\partial^2}{\partial t^2} \frac{1}{c^2} E - \frac{2\mu\sigma}{c} \frac{\partial}{\partial t} E \qquad (9)$$

where

c = velocity of light

μ = permeability of gain medium

σ = conductivity of gain medium

$\mu\sigma$ = G the gain coefficient of the medium

Since the field consists of right and left-traveling waves, these traveling waves individually satisfy the Helmholtz equation. Substituting the right-traveling wave from Eq.(6) into Eq.(9) yields

$$2ik\left(\frac{\partial}{\partial z} f(z,x) - Gf(z,x)\right) + \frac{\partial^2}{\partial x^2} f(z,x) = 0 \qquad (10)$$

when second derivatives of the z-coordinate are neglected due to the slow variation of the field as a function of z.

Similarly, when the left-traveling wave of Eq.(6) is substituted into Eq.(9)

$$2ik\left(\frac{\partial}{\partial \rho} g(\rho,\theta) - Gg(\rho,\theta)\right) + \rho^{-2} \frac{\partial^2}{\partial \theta^2} g(\rho,\theta) = 0 \qquad (11)$$

where second derivatives with respect to ρ are neglected due to the slow variation of the field as a function of ρ. Also terms of order $\frac{1}{4}\rho^{-\frac{5}{2}} g(\rho,\theta)$ are neglected when compared to

$2k\rho^{-\frac{1}{2}} \frac{\partial}{\partial\rho} g(\rho,\theta)$. In Eq.(10) and Eq.(11) the G term is assumed to be a constant.

If diffraction effects are ignored the second derivative terms of Eq. (10) and Eq.(11) can be neglected (Ref 6:229) yielding the geometrical rate equations

$$\frac{\partial}{\partial z}f(z) = Gf(z) \tag{12}$$

$$\frac{\partial}{\partial \rho}g(\rho) = Gg(\rho) \tag{13}$$

The solutions of Eq.(12) and Eq.(13) are

$$f(z) = f(0)\exp\{Gz\} \tag{14}$$

$$g(\rho) = g(0)\exp\{G\rho\} \tag{15}$$

The right and left-traveling waves in the cavity are dependent upon each other. This dependence can be expressed in terms of the gain in the cavity. The Wronskian of Eq.(12) and Eq.(13) can be found by multiplying Eq.(12) by $g(\rho)$ and Eq. (13) by $f(z)$ and then subtracting one equation from the other. In the paraxial approximation, the ρ dependence is changed to z using Eq.(4). This yields

$$\left[\frac{\partial}{\partial z}f(z)\right]g(Md-z) = \left[-\frac{\partial}{\partial z}g(Md-z)\right]f(z) \tag{16}$$

Integrating yields

$$f(z)g(Md-z) = C \tag{17}$$

where C is a constant of integration.

Evaluating this expression at each mirror yields

$$f(0)g(Md) = f(D)g(d) \tag{18}$$

Upon substituting for $f(0)$ from Eq.(7) and for $f(D)$ from Eq.(8) it is found that

$$e^{2ikD}M^{-\frac{1}{2}}g^2(Md) = g^2(d) \tag{19}$$

From Eq.(15)

$$g^2(Md) = g^2(d)e^{2GD} \tag{20}$$

A constant gain coefficient Q which satisfies the steady state requirement for this analysis can be defined as

$$Q = GD \tag{21}$$

Then substituting Eq.(20) into Eq.(19) yields

$$\mu = M^{-\frac{1}{2}}e^{2ikD}e^{2Q} \tag{22}$$

where, for this geometrical case

$$\mu = 1 \tag{23}$$

The μ terms are the eigenvalue for the resonator modes. When diffraction effects are included, the μ terms are found to be the roots of a polynomial equation where each root is the eigenvalue of a different resonant mode. Thus

the constant gain required to sustain each mode in steady state is different.

Diffraction Effects

The amplitude functions of the right and left-traveling waves are expanded in terms of a geometrical amplitude function and edge diffraction amplitude functions as shown by Horwitz (Ref 3:1530) and Moore and McCarthy (Ref 6:230). The diffraction terms consist of slowly varying amplitude functions and phase terms that have rapid transverse oscillations. Well within the geometrical shadow boundaries the diffraction terms can be viewed as cylindrical wavelets emanating from the virtual images of the feedback mirror edges (Ref 6:230). From Appendix B, these virtual images are located on two half-parabolas. For the (d,b) edge of the feedback mirror, the images are located at

$$(\rho_n, b_n) = (dM^{2n}, bM^n) \; ; n=0,\pm 1,\pm 2,\ldots,\pm N \tag{24}$$

and for the $(d,-a)$ edge

$$(\rho_n, -a_n) = (dM^{2n}, -aM^n) ; n=0,\pm 1,\pm 2,\ldots,\pm N \tag{25}$$

For $n>0$ these images lie behind the large mirror and their cylindrical wavelets contribute to the right-traveling wave amplitude function which can be written as

$$f(z,x) = \left[\sum_{n=1}^{N} (\rho_n-\rho)^{-\frac{1}{2}} r_n(\rho,x) \exp\left\{\frac{ik}{2}\,\frac{(x-b_n)^2}{(\rho_n-\rho)}\right\}\right.$$

$$\left. +(\rho_n-\rho)^{-\frac{1}{2}} s_n(\rho,x) \exp\left\{\frac{ik}{2}\,\frac{(x+a_n)^2}{(\rho_n-\rho)}\right\}\right] \qquad (26)$$

$$+\hat{f}(\rho,x)$$

where

$r_n(\rho,x)$ = diffraction amplitude functions from left images of edge (d,b)

$s_n(\rho,x)$ = diffraction amplitude functions from left images of edge (d,-a)

$\hat{f}(\rho,x)$ = geometrical amplitude function

The $n=0$ wavelets originate at the feedback mirror edges and the $n<0$ wavelets originate at the virtual images to the right of the feedback mirror. These images approach the focal point as $n\to-\infty$. These cylindrical wavelets contribute to the left-traveling amplitude function which takes the form

$$\rho^{-\frac{1}{2}} g(\rho,\theta) = \left[\sum_{n=0,-1,-2}^{-N+1} (\rho-\rho_n)^{-\frac{1}{2}} u_n(\rho,x) \exp\left\{\frac{ik}{2}\,\frac{\left(\theta-\frac{b_n}{\rho_n}\right)^2}{\frac{1}{\rho_n}-\frac{1}{\rho}}\right\}\right.$$

$$\left. +(\rho-\rho_n)^{-\frac{1}{2}} v_n(\rho,x) \exp\left\{\frac{ik}{2}\,\frac{\left(\theta+\frac{a_n}{\rho_n}\right)^2}{\frac{1}{\rho_n}-\frac{1}{\rho}}\right\}\right] \qquad (27)$$

$$+\rho^{-\frac{1}{2}} \hat{g}(\rho,x)$$

where

$u_n(\rho,x)$ = diffraction amplitude function from right images of edge (d,b)

$v_n(\rho,x)$ = diffraction amplitude function from right images of edge $(d,-a)$

$\hat{g}(\rho,x)$ = geometrical amplitude function

The phases appearing in Eq.(26) and Eq.(27) are the Fresnel approximations to the phases of cylindrical waves emanating from points (ρ_n, b_n) or $(\rho_n, -a_n)$. When $|n|$ is small the phase terms are rapidly oscillating but when $|n|$ becomes large the phases associated with edge (d,b) terms approach the constant phase

$$\exp\left\{\frac{ikb^2}{2d}\right\} = \exp\{2\pi i F_{eb}\} \tag{28}$$

and the phases associated with edge $(d,-a)$ terms approach the constant phase

$$\exp\left\{\frac{ika^2}{2d}\right\} = \exp\{2\pi i F_{ea}\} \tag{29}$$

It has been found by Horwitz (Ref 3:1533) that a practical value for large N is given by

$$M^N = 250F_{eff} \tag{30}$$

This is practical in the sense that a larger N changes the eigenvalues at most in the third decimal place. In the

off-center case let $F_{eff}=F_{eb}$ since $F_{eb}>F_{ea}$. This will assure a large enough number of terms in the expansion.

Finally, substituting Eq.(26) and Eq.(27) into Eq.(9) yields the rate equations

$$\frac{\partial r_n}{\partial s} = Gr_n \ ; \ \frac{\partial s_n}{\partial s} = Gs_n \ ; \ \frac{\partial \hat{f}}{\partial z} = G\hat{f} \tag{31}$$

$$\frac{\partial u_n}{\partial s} = Gu_n \ ; \ \frac{\partial v_n}{\partial s} = Gv_n \ ; \ \frac{\partial \hat{g}}{\partial z} = G\hat{g} \tag{32}$$

where $\frac{\partial}{\partial s}$ is the directional derivative along the direction of the diffraction ray propagation off the optical axis. This is consistent with previous gain considerations.

In the special case where $|a| = |b|$ these preceding expressions reduce to those of Moore and McCarthy (Ref 6) for the centered unstable confocal resonator.

In summary, this section has developed the expansions for the right and left-traveling wave amplitude functions in terms of geometric and edge diffraction amplitude functions.

III. Boundary Conditions on the Mirrors

This chapter details the development of the boundary conditions, including the diffraction terms, on the two mirrors of the cavity. These conditions specify the relationships between the right and left-traveling waves. The development is arranged into two sections. The boundary conditions on the large mirror are developed first. The second section develops the boundary conditions on the feedback mirror using an asymptotic expansion of a diffraction integral.

Boundary Conditions on the Large Mirror

Since the large mirror is assumed to reflect all of the incident field, the boundary conditions on it are easily obtained. From Eq.(7) the boundary conditions on the large mirror are found be equating the right hand sides of Eqs.(26) and (27) for ρ=Md and θ=x/Md . Using Eq.(24) it can be shown that the resulting equation has like exponentials on each side of the equality

$$\exp\left\{\frac{ik}{2}\,\frac{(x-b_n)^2}{(\rho_n-Md)}\right\} = \exp\left\{\frac{ik}{2}\,\frac{\left(\frac{x}{Md}-\frac{b_{1-n}}{\rho_{1-n}}\right)^2}{\left(\frac{1}{\rho_{1-n}}-\frac{1}{Md}\right)}\right\}_{n=1,2...,N} \qquad (33)$$

$$\exp\left\{\frac{ik}{2}\,\frac{(x+a_n)^2}{(\rho_n-Md)}\right\} = \exp\left\{\frac{ik}{2}\,\frac{\left(\frac{x}{Md}+\frac{a_{1-n}}{\rho_{1-n}}\right)^2}{\left(\frac{1}{\rho_{1-n}}-\frac{1}{Md}\right)}\right\}_{n=1,2...,N} \qquad (34)$$

The coefficients of these like exponentials must also be equal. Equating these coefficients yields the boundary conditions for the diffraction amplitudes:

$$(\rho_n - Md)^{-\frac{1}{2}} r_n(Md,x) = (Md - \rho_{1-n})^{-\frac{1}{2}} u_{1-n}(Md,x) \qquad ;n=1,2,\ldots,N \tag{35}$$

$$(\rho_n - Md)^{-\frac{1}{2}} s_n(Md,x) = (Md - \rho_{1-n})^{-\frac{1}{2}} v_{1-n}(Md,x) \qquad ;n=1,2,\ldots,N \tag{36}$$

Equating the remaining terms yields the boundary condition for the geometrical amplitudes:

$$\hat{f}(Md,x) = (Md)^{-\frac{1}{2}} \hat{g}(Md,x) \tag{37}$$

<u>Boundary Conditions on the Feedback Mirror</u>

The boundary conditions on the feedback mirror are more difficult to develop. While the total field must vanish on the mirror, the left-traveling component must be zero off the mirror. A diffraction integral evaluated over the feedback mirror will account for these requirements. It is assumed that the gain coefficient is zero very near the feedback mirror, so this term is not included in the integration. Then, the Huygen-Fresnel diffraction integral will give an expression for $g(\rho,\theta)$ in terms of $g(d,\theta)$:

$$g(\rho,\theta) = \frac{1}{2\pi}\left[\frac{2\pi ik}{\frac{1}{\rho} - \frac{1}{d}}\right]^{\frac{1}{2}} \int_{-a/d}^{b/d} g(d,\theta') \tag{38}$$

$$\times \exp\left\{\frac{-ik}{2}\frac{(\theta-\theta')^2}{\frac{1}{\rho}-\frac{1}{d}}\right\} d\theta'$$

where the Fresnel degree of approximation has been used in phase terms. Making the change of variables, $\eta=\theta'\frac{d}{a}$, and replacing $g(d,\theta')$ with the form from Eq.(8) yields

$$g(\rho,\theta) = \frac{1}{2\pi}\left[\frac{2\pi ika^2}{d\left(\frac{1}{\rho}-\frac{1}{d}\right)}\right]^{\frac{1}{2}} e^{2ikD}$$

$$\times \int_{-1}^{b/a} f(D,a\eta)\exp\left\{\frac{-ik}{2}\frac{\left(\theta-\frac{a}{d}\eta\right)^2}{\frac{1}{\rho}-\frac{1}{d}}\right\} d\eta \tag{39}$$

The expression for $f(D,a\eta)$ given by Eq.(26) is substituted into this equation. This integral can be approximated by an asymptotic expansion about the stationary phase point as described in Appendix C. The results of the specific integration in Eq.(39) are developed in Appendix D where the approximation neglects terms of higher order than F_{ea}^{-1}. The expression for $g(\rho,\theta)$ near the feedback mirror is then found to be

$$g(\rho,\theta) = \frac{1}{2\pi}\left[\frac{2\pi i k a^2}{d\left(\frac{1}{\rho}-\frac{1}{d}\right)}\right]^{\frac{1}{2}} e^{2ikD} \tag{40}$$

$$\times \left[I_1+I_2+I_3\right]$$

where I_1, I_2, and I_3 are developed in Appendix D. Eq.(D-43) of this appendix presents Eq.(40) in detail.

Recalling Eq.(27) and letting the summation index $n = -m$ yields

$$g(\rho,\theta) = \left[\sum_{m=0,1,2}^{N-1} \rho^{\frac{1}{2}}(\rho-\rho_{-m})^{-\frac{1}{2}}\right.$$

$$\times \left[u_{-m}(\rho,x)\exp\left\{\frac{ik}{2}\frac{\left(\theta-\frac{b_{-m}}{\rho_{-m}}\right)^2}{\left(\frac{1}{\rho_{-m}}-\frac{1}{\rho}\right)}\right\}\right. \tag{41}$$

$$\left.\left.+v_{-m}(\rho,x)\exp\left\{\frac{ik}{2}\frac{\left(\theta+\frac{a_{-m}}{\rho_{-m}}\right)^2}{\left(\frac{1}{\rho_{-m}}-\frac{1}{\rho}\right)}\right\}\right]\right]+\hat{g}(\rho,x)$$

The expressions for $g(\rho,\theta)$ in Eq.(41) and Eq.(40) can then be equated. It can be shown that as ρ approaches d there are like exponentials on both sides of the equality

$$\exp\left\{\frac{ik}{2}\frac{\rho(b_n-\theta d)^2}{\rho(\rho_n-d)+d(\rho-d)}\right\} \Rightarrow \exp\left\{\frac{ik}{2}\frac{\left(\theta+\frac{b_{-n}}{\rho_{-n}}\right)^2}{\left(\frac{1}{\rho_{-n}}-\frac{1}{\rho}\right)}\right\} \tag{42}$$

$$n = 1,2\ldots,N-1$$

and

$$\exp\left\{\frac{ik}{2}\,\frac{\rho(a_n+\theta d)^2}{\rho(\rho_n-d)+d(\rho-d)}\right\} \Rightarrow \exp\left\{\frac{ik}{2}\,\frac{\left(\theta-\dfrac{a_{-n}}{\rho_{-n}}\right)^2}{\left(\dfrac{1}{\rho_{-n}}-\dfrac{1}{\rho}\right)}\right\} \tag{43}$$

$$n = 1,2,\ldots,N-1$$

Therefore the coefficients of these like exponentials can be equated as ρ approaches d . These are the boundary conditions for the diffraction amplitude functions on the feedback mirror:

$$e^{2ikD}(\rho_n-d)^{-\frac{1}{2}}r_n(d,x) = (d-\rho_{-n})^{-\frac{1}{2}}u_{-n}(d,x) \tag{44}$$

$$n = 1,2,\ldots,N-1$$

$$e^{2ikD}(\rho_n-d)^{-\frac{1}{2}}s_n(d,x) = (d-\rho_{-n})^{-\frac{1}{2}}v_{-n}(d,x) \tag{45}$$

$$n = 1,2,\ldots,N-1$$

The geometrical term from Eq.(41) can then be equated to the geometrical term and the Nth diffraction terms. Here N is assumed large as given by Eq.(30) and so the phases are constant as given by Eq.(28) and Eq.(29). The boundary condition for the geometrical terms as ρ approaches d is then

$$d^{-\frac{1}{2}}\hat{g}(d,x) = e^{2ikD}\left[(\rho_N-d)^{-\frac{1}{2}}r_N(d,x)\exp\left\{\frac{ikb^2}{2d}\right\}\right. \tag{46}$$

$$\left.+(\rho_N-d)^{-\frac{1}{2}}s_N(d,x)\exp\left\{\frac{ika^2}{2d}\right\}+\hat{f}(d,x)\right]$$

Then the $m=0$ terms remain from Eq.(41) which equal the remaining terms from Eq.(40). This is taken as the expression for the diffraction amplitude functions on their first trip across the cavity and is used in the next chapter.

In summary, this chapter has developed the relationship between the various amplitude functions at the mirror boundaries. It remains to relate these amplitude functions to a common expression, the diffraction amplitudes after one round trip in the cavity.

IV. Amplitude Functions in the Resonator

This chapter develops the relationships among the geometric and the diffraction amplitude functions. The development of these relationships consists of three sections. The first section relates the diffraction amplitude functions on the nth round trip of the cavity (equivalent to emanating from the nth virtual image of the feedback mirror edges) to the first round trip value. The first round trip amplitudes are then defined in terms of auxiliary functions, $q_a(x)$ and $q_b(x)$. The second section relates the geometrical amplitude functions to these auxiliary functions. In the last section the expressions for these auxiliary functions are developed in terms of the eigenvalues for the resonator modes.

It is important to determine these relationships between the various amplitude functions because the values of these amplitudes can be found once the value for the first round trip diffraction amplitude functions are known. With values for all the amplitude functions the total amplitude functions for the right and left-traveling wave can be determined.

Diffraction Amplitude Functions

Initial Amplitude Functions. After the boundary conditions on the feedback mirror are extracted from the right hand side of Eq.(41) equated with the right hand side of

Eq.(40), the remaining terms yield expressions for the diffraction amplitude functions on their first trip across the cavity from the feedback mirror. From these remaining terms it can be shown that the following exponentials are equal by using Eq.(24) and Eq.(25):

$$\exp\left\{\frac{ik}{2}\,\frac{\left(\theta-\frac{b_0}{\rho_0}\right)^2}{\frac{1}{\rho_0}-\frac{1}{\rho}}\right\} = \exp\left\{\frac{ik}{2}\,\frac{\rho(b-\theta d)^2}{d(\rho-d)}\right\} \tag{47}$$

$$\exp\left\{\frac{ik}{2}\,\frac{\left(\theta+\frac{a_0}{\rho_0}\right)^2}{\frac{1}{\rho_0}-\frac{1}{\rho}}\right\} = \exp\left\{\frac{ik}{2}\,\frac{\rho(a+\theta d)^2}{d(\rho-d)}\right\} \tag{48}$$

This leads to the expressions for the diffraction amplitude functions on their first trip across the cavity:

$$\begin{aligned} u_0(\rho,x) = \mp\left(\frac{i}{2\pi k}\right)^{\frac{1}{2}} e^{2ikD}\Bigg[\sum_{n=1}^{N}(\rho_n-d)^{-\frac{1}{2}} & \qquad (49)\\ \times\, r_n(d,b)\exp\left\{\frac{ik}{2}\,\frac{(b-b_n)^2}{\rho_n-d}\right\}\left[\frac{b-b_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right]^{-1} & \\ +(\rho_n-d)^{-\frac{1}{2}}s_n(d,b)\exp\left\{\frac{ik}{2}\,\frac{(b+a_n)^2}{\rho_n-d}\right\} & \\ \times\left[\frac{b+a_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right]^{-1}+\hat{f}(d,b)\left[\frac{\rho(b-\theta d)}{d(\rho-d)}\right]^{-1}\Bigg] & \end{aligned}$$

and

$$v_0(\rho,x) = \mp\left(\frac{i}{2\pi k}\right)^{\frac{1}{2}} e^{2ikD}\left[\sum_{n=1}^{N}(\rho_n-d)^{-\frac{1}{2}} r_n(d,-a)\right.$$

$$\times \exp\left\{\frac{ik}{2}\frac{(a+b_n)^2}{(\rho_n-d)}\right\}\left[\frac{a+b_n}{\rho_n-d}+\frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1} \qquad (50)$$

$$\times(\rho_n-d)^{-\frac{1}{2}} s_n(d,-a)\exp\left\{\frac{ik}{2}\frac{(a-a_n)^2}{\rho_n-d}\right\}$$

$$\left.\times\left[\frac{a-a_n}{\rho_n-d}+\frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1}+\hat{f}(d,-a)\left[\frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1}\right]$$

The $\mp$ at the beginning of the righthand side of Eqs.(49) and (50) explicitly denotes the choice of roots from the square root. In Eq.(40) the positive root of the square root was implicitly chosen so that the real part was positive (Ref6:231). This leads to the choice of the negative root here.

In Eq.(49) it can be shown that

$$\left[\frac{b-b_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right]^{-1} = \left[\beta_{bn}-\beta\right]^{-1} \qquad (51)$$

and

$$\left[\frac{b+a_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right]^{-1} = \left[\beta_{an}-\beta\right]^{-1} \qquad (52)$$

where

$$\beta = \left(\frac{x-b}{\rho-d}\right) \qquad (53)$$

is the angle with respect to the optical axis of the ray joining the mirror edge (d,b) with a point (ρ,x) in the cavity,

$$\beta_{bn} = \left(\frac{b-b_n}{\rho_n-d} + \frac{b}{d}\right) = \frac{b}{d}(1+M^{-n})^{-1} \tag{54}$$

is the angle for the ray joining image point (ρ_{-n}, b_{-n}) with edge (d,b),

$$\beta_{an} = \left(\frac{b+a_n}{\rho_n-d} + \frac{b}{d}\right) = \frac{b}{d}\left(\frac{1+\frac{a}{b}M^{-n}}{1-M^{-2n}}\right) \tag{55}$$

is the angle of the ray joining points $(\rho_{-n}, -a_{-n})$ with edge (d,b) . These angles are illustrated in Fig.(2). Also it can be shown that

$$\left[\frac{\rho(b-\theta d)}{d(\rho-d)}\right]^{-1} = \left[\frac{b}{d} - \beta\right]^{-1} \tag{56}$$

Similarly, for Eq.(50)

$$\left[\frac{a+b_n}{\rho_n-d} + \frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1} = \left[\alpha_{bn}+\alpha\right]^{-1} \tag{57}$$

and

$$\left[\frac{a-a_n}{\rho_n-d} + \frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1} = \left[\alpha_{an}+\alpha\right]^{-1} \tag{58}$$

where

$$\alpha = \left(\frac{x+a}{\rho-d}\right) \tag{59}$$

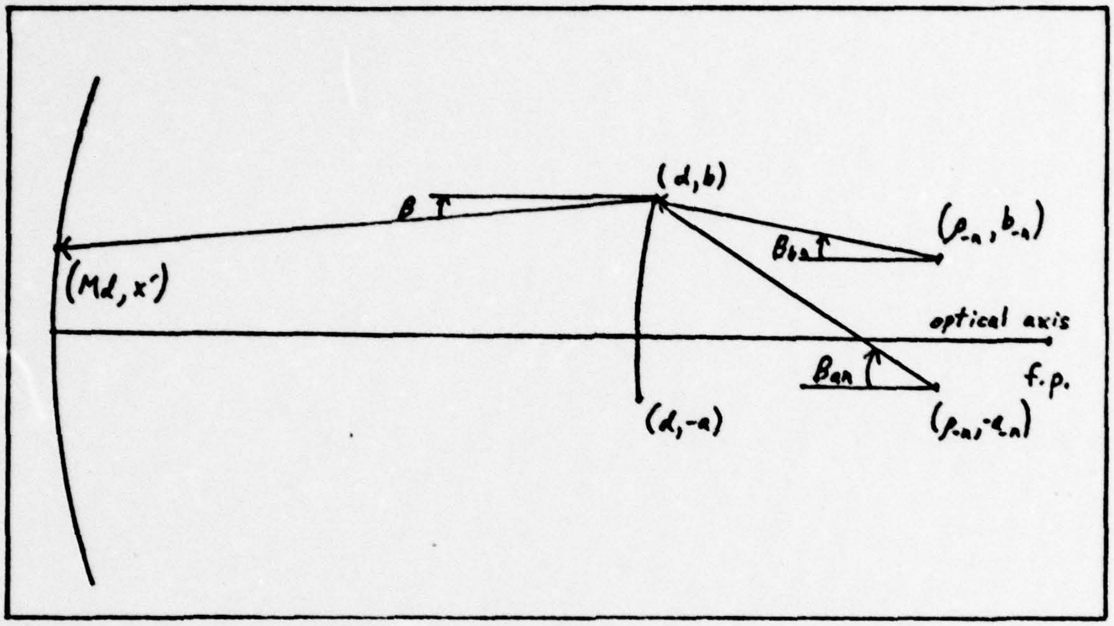

Fig 2 RAY ANGLES FOR $u_0(\rho,x)$ AMPLITUDE FUNCTION

is the angle of the ray joining edge (d,-a) with a point (ρ,x) in the cavity,

$$\alpha_{bn} = \left(\frac{a+b_n}{\rho_n-d} + \frac{a}{d}\right) = \frac{a}{d}\left(\frac{1+\frac{b}{a}M^{-n}}{1-M^{-2n}}\right) \qquad (60)$$

is the angle of the ray joining points (ρ_{-n},b_{-n}) with edge (d,-a),

$$\alpha_{an} = \left(\frac{a-a_n}{\rho_n-d} + \frac{a}{d}\right) = \frac{a}{d}\left[1+M^{-n}\right]^{-1} \qquad (61)$$

is the angle of the line joining points (ρ_{-n},$-a_{-n}$) with edge (d,-a) . These angles are illustrated in Fig.(3). Also it can be shown that

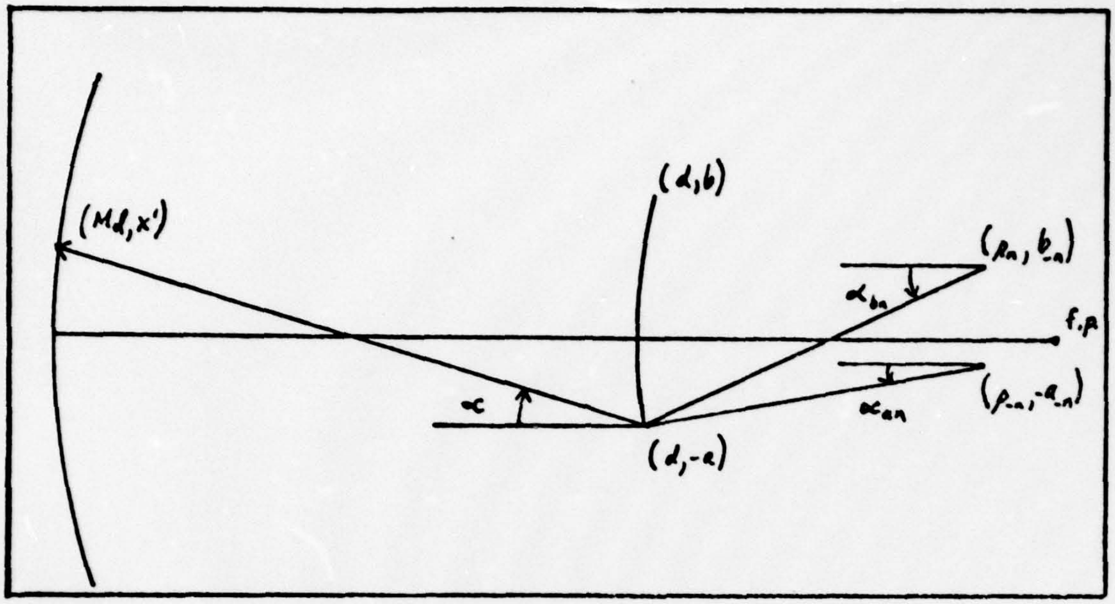

Fig 3 RAY ANGLES FOR $v_0(\rho,x)$ AMPLITUDE FUNCTION

$$\left[\frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1} = \left[\frac{a}{d} + \alpha\right]^{-1} \tag{62}$$

It can also be shown that using the angle expressions, α and β , allows the left-hand sides of Eq.(49) and Eq.(50) to be written as

$$u_0(\rho,x) = u_0(\rho,(\rho-d)\beta+b) \tag{63}$$

$$v_0(\rho,x) = v_0(\rho,(\rho-d)\alpha-a) \tag{64}$$

Recall that Eq.(49) and Eq.(50) are valid only near the feedback mirror because the gain coefficient was assumed to be zero when these equations were developed. To propagate across the cavity, maintaining steady state, the diffraction

amplitudes of Eq.(49) and Eq.(50) must be multiplied by the exponential of the gain coefficient, Q , from Eq.(21). Then, using Eq.(51) through Eq.(64), the diffracted amplitude functions on the first trip across the resonator can be expressed:

$$u_0(\rho,(\rho-d)\beta+b) = -\left(\frac{i}{2\pi k}\right)^{\frac{1}{2}} e^{Q} e^{2ikD}$$

$$\times\left[\sum_{n=1}^{N}(\rho_n-d)^{-\frac{1}{2}} r_n(d,b)\right.$$

$$\times \exp\left\{\frac{ik}{2}\,\frac{(b-b_n)^2}{\rho_n-d}\right\}\left[\beta_{bn}-\beta\right]^{-1} \tag{65}$$

$$+(\rho_n-d)^{-\frac{1}{2}} s_n(d,b)\exp\left\{\frac{ik}{2}\,\frac{(b+a_n)^2}{\rho_n-d}\right\}$$

$$\times\left[\beta_{an}-\beta\right]^{-1}$$

$$\left.+\hat{f}(d,b)\left[\frac{b}{d}-\beta\right]^{-1}\right]$$

and

$$v_0(\rho,(\rho-d)\alpha-a) = -\left(\frac{i}{2\pi k}\right)^{\frac{1}{2}} e^{Q} e^{2ikD}$$

$$\times\left[\sum_{n=1}^{N}(\rho_n-d)^{-\frac{1}{2}} r_n(d,-a)\right.$$

$$\times \exp\left\{\frac{ik}{2}\,\frac{(a+b_n)^2}{\rho_n-d}\right\}\left[\alpha_{bn}+\alpha\right]^{-1} \tag{66}$$

$$+(\rho_n-d)^{-\frac{1}{2}}s_n(d,-a)\exp\left\{\frac{ik}{2}\frac{(a-a_n)^2}{\rho_n-d}\right\}$$

$$\times\left[\left[\alpha_{an}+\alpha\right]^{-1}+\hat{f}(d,-a)\left[\frac{a}{\alpha}+\alpha\right]^{-1}\right] \quad (66)$$

Propagation Equations Across Cavity. An expression is needed to relate the amplitude functions of the right-traveling diffraction terms at the feedback mirror on the nth round-trip to their values for the first round-trip of the cavity. To develop these round-trip expressions, the propagation expressions are needed for diffracted rays going from the feedback mirror to the large mirror on the nth trip to the left and a similar expression for the nth trip to the right. On the nth trip to the left or right a diffracted ray can be viewed as originating from the corresponding nth virtual image of the appropriate feedback mirror edge. The diffracted wave on the nth round trip travels that part of the ray path inside the cavity. This is illustrated in Fig.(4).

For these image rays to the right, intersecting the large mirror at any point (Md,x), the amplitude functions along the ray paths, including steady state gain, are given by

$$r_n\left(Md-z,x-\frac{x-b_n}{Md-\rho_n}z\right)=r_n(Md,x)e^Q \quad (67)$$

$$n=1,2,\ldots,N$$

$$s_n\left(Md-z,x-\frac{x+a_n}{Md-\rho_n}z\right)=s_n(Md,x)e^Q \quad (68)$$

$$n=1,2,\ldots,N$$

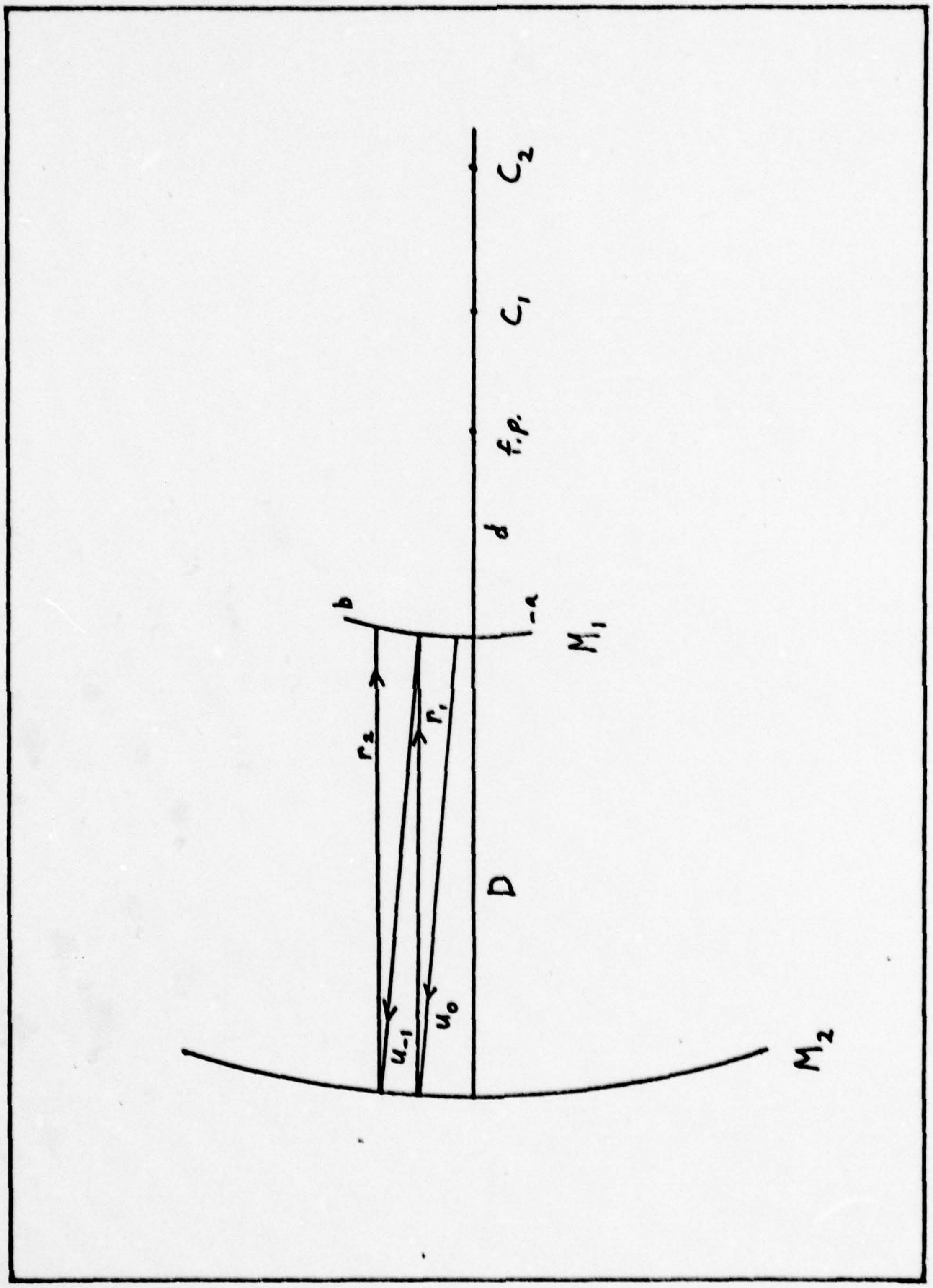

Fig 4 RAY PATHS IN RESONATOR

Similarly, for the nth ray paths to the left, intersecting the feedback mirror at any point (d,x) , yields

$$u_n(\rho, x+\frac{x-b_n}{d-\rho_n}(\rho-d)) = u_n(d,x)e^Q \tag{69}$$

$$n = -1,-2,\ldots,-N+1$$

$$v_n(\rho, x+\frac{x+a_n}{d-\rho_n}(\rho-d)) = v_n(d,x)e^Q \tag{70}$$

$$n = -1,-2,\ldots,-N+1$$

Amplitude Functions from nth Transit of Cavity. The expressions relating r_n to r_1 and s_n to s_1 will now be developed. The diffracted amplitude function $r_n(d,x)$ is the amplitude of the diffracted wave caused by the feedback mirror edge (d,b) after it has made n round trips through the cavity. This function can be related to the $r_{n-1}(d,x')$ amplitude function by propagating it back through the cavity one round trip. Here (d,x') is the point a ray must start from on the feedback mirror to propagate to point (d,x) . .

Starting with $r_n(d,x)$ and using Eq.(67) to propagate the amplitude function on the feedback mirror to the large mirror yields

$$r_n(d,x) = r_n(Md,\eta)e^Q \qquad n = 1,2,\ldots,N \tag{71}$$

where, using Eq.(1) and Eq.(24)

$$\eta = x \frac{M-M^{2n}}{1-M^{2n}} - b \frac{M^{n}(M-1)}{1-M^{2n}} \tag{72}$$

At the large mirror the boundary condition of Eq.(35) gives

$$e^{Q} r_{n}(Md,\eta) = e^{Q} \left(\frac{M^{2n}-M}{M-M^{2(1-n)}}\right)^{\frac{1}{2}} u_{1-n}(Md,\eta) \tag{73}$$

$$n = 1,2,\ldots,N$$

From this Eq.(69) is used to propagate $u_{1-n}(Md,\eta)$ back to the feedback mirror:

$$e^{Q}\left(\frac{M^{2n}-M}{M-M^{2(1-n)}}\right)^{\frac{1}{2}} u_{1-n}(Md,\eta)$$

$$= e^{2Q}\left(\frac{M^{2n}-M}{M-M^{2(1-n)}}\right)^{\frac{1}{2}} u_{1-n}(d,x') \tag{74}$$

$$n = 2,3,\ldots,N$$

where

$$x' = \eta \frac{1-M^{2(1-n)}}{M-M^{2(1-n)}} + b \frac{M^{(1-n)}(M-1)}{M-M^{2(1-n)}} \tag{75}$$

The boundary condition at the feedback mirror Eq.(44) then yields

$$e^{2Q}\left(\frac{M^{2n}-M}{M-M^{2(1-n)}}\right)^{\frac{1}{2}} u_{1-n}(d,x')$$

$$= e^{2Q} e^{2ikD} r_{n-1}(d,x') \tag{76}$$

$$\times\left(\frac{1-M^{2(1-n)}}{M^{2(n-1)}-1}\;\frac{M^{2n}-M}{M-M^{2(1-n)}}\right)^{\frac{1}{2}}$$

$$n = 2,3,\ldots,N$$

Using Eq.(72) and Eq.(75), the x' value can be expressed in terms of x. Then the r_n amplitude function can be related to the r_{n-1} function using Eq.(71), Eq.(73), Eq.(74) and Eq.(76):

$$r_n(d,x) = M^{\frac{1}{2}} e^{2Q} e^{2ikD}$$

$$\times\; r_{n-1}\left(d, x\frac{M^{2n}-M}{M^{2n}-1} + b\frac{M^{n}(M-M^{-1})}{M^{2n}-1}\right) \tag{77}$$

$$n = 2,3,\ldots,N$$

From this equation r_{n-1} can be related to r_{n-2} which can then also be related back to r_n. Continuing in this manner leads to an expression relating r_n to r_1:

$$r_n(d,x) = M^{\frac{n-1}{2}} e^{2(n-1)Q} e^{2ikD(n-1)} \tag{78}$$

$$\times\; r_1(d,k_{bn}(x)) \;;\; n=1,2,\ldots N$$

where

$$k_{bn}(x) = x\frac{M^{n-1}(M^2-1)}{M^{2n}-1} + b\frac{M^{2n-1}-M}{M^{2n}-1} \tag{79}$$

By a similar procedure the diffracted amplitude function s_n can be related to s_1 which yields

$$s_n(d,x) = M^{\frac{n-1}{2}} e^{2(n-1)Q} e^{2ikD(n-1)} \tag{80}$$

$$X \; s_1(d,k_{an}(x)) \; ; \; n=1,2,\ldots N$$

where

$$k_{an}(x) = x \frac{M^{n-1}(M^2-1)}{M^{2n}-1} - a \frac{M^{2n-1}-M}{M^{2n}-1} \tag{81}$$

<u>First Round Trip Amplitude Function</u>. From Eq.(78) and Eq.(80) the nth round trip diffracted amplitude functions are related to that of the first round trip. Expressions are now developed for the first round trip amplitudes.

From Eq.(65) the diffracted amplitude u_0 is known at any point (Md,x') on the large mirror by letting $\rho=Md$ and substituting x' into Eq.(53) for angle β . This gives the amplitude function

$$u_0(Md,x') = u_0(Md,b+(Md-d)\beta) \tag{82}$$

where

$$\beta = \frac{x'-b}{d(M-1)} \tag{83}$$

The boundary condition at the large mirror Eq.(35) is then used to relate u_0 to r_1 :

$$u_0(Md,x') = \left(\frac{Md-\rho_0}{\rho_1-Md}\right)^{\frac{1}{2}} r_1(Md,x') \tag{84}$$

Then Eq.(67) is used to propagate $r_1(Md,x')$ across the cavity to $r_1(d,x)$:

$$e^{Q} r_1(Md,x') = r_1(d,x) \tag{85}$$

Therefore, combining Eq.(84) and Eq.(85) yields

$$r_1(d,x) = M^{\frac{1}{2}} e^{Q} u_0(Md,x') \tag{86}$$

where

$$x' = (x+b)\frac{M}{1+M} \tag{87}$$

Also, from Eq.(83)

$$\beta = \frac{xM^{-1}-bM^{-2}}{d(1-M^{-2})} \tag{88}$$

Finally, using Eq.(86), Eq.(88), Eq.(54), and Eq.(55) in Eq.(65) yields the needed diffraction amplitude expression, $r_1(d,x)$,

$$r_1(d,x) = -M^{\frac{1}{2}} e^{2Q} e^{2ikD} \left(\frac{i}{2\pi k}\right)^{\frac{1}{2}} \tag{89}$$

$$X \left[\sum_{n=1}^{N} d^{-\frac{1}{2}} (M^{2n}-1)^{-\frac{1}{2}} \exp\left\{\frac{ikb^2}{2d} \frac{(1-M^n)^2}{M^{2n}-1}\right\}\right.$$

$$\left. X \;\; r_n(d,b) \left[\frac{b}{d}\,\frac{1-M^{-n}}{1-M^{-2n}} - \frac{xM^{-1}-bM^{-2}}{d(1-M^{-2})}\right]^{-1}\right]$$

$$+d^{-\frac{1}{2}}(M^{2n}-1)^{-\frac{1}{2}}\exp\left\{\frac{ika^2}{2d}\,\frac{\left(\frac{b}{a}+M^n\right)^2}{M^{2n}-1}\right\} \tag{89}$$

$$Xs_n(d,b)\left[\frac{a}{d}\,\frac{\frac{b}{a}+M^{-n}}{1-M^{-2n}} - \frac{xM^{-1}-bM^{-2}}{d(1-M^{-2})}\right]^{-1}$$

$$\left.+\hat{f}(d,b)\left[\frac{b}{d} - \frac{xM^{-1}-bM^{-2}}{d(1-M^{-2})}\right]^{-1}\right]$$

Similarly, the amplitude function $s_1(d,x)$ can be developed:

$$s_1(d,x) = -M^{\frac{1}{2}}e^{2Q}e^{2ikD}\left(\frac{i}{2\pi k}\right)^{\frac{1}{2}}$$

$$X\left[\sum_{n=1}^{N} d^{-\frac{1}{2}}(M^{2n}-1)^{-\frac{1}{2}}\exp\left\{\frac{ikb^2}{2d}\,\frac{\left(\frac{a}{b}+M^n\right)^2}{M^{2n}-1}\right\}\right.$$

$$Xr_n(d,-a)\left[\frac{b}{d}\,\frac{\frac{a}{b}+M^{-n}}{1-M^{-2n}} + \frac{xM^{-1}+aM^{-2}}{d(1-M^{-2})}\right]^{-1}$$

$$+d^{-\frac{1}{2}}(M^{2n}-1)^{-\frac{1}{2}}\exp\left\{\frac{ika^2}{2d}\,\frac{(1-M^n)^2}{M^{2n}-1}\right\} \tag{90}$$

$$Xs_n(d,-a)\left[\frac{a}{d}\,\frac{1-M^{-n}}{1-M^{-2n}} + \frac{xM^{-1}+aM^{-2}}{d(1-M^{-2})}\right]^{-1}$$

$$\left.+\hat{f}(d,-a)\left[\frac{a}{d} + \frac{xM^{-1}+aM^{-2}}{d(1-M^{-2})}\right]^{-1}\right]$$

Up to this point the diffraction amplitude functions, $u_0(\rho,\theta)$ and $v_0(\rho,\theta)$, have been propagated through one round trip in the cavity. Also, all of the succeeding round trip amplitude functions have been expressed in terms of these first round trip amplitudes (Eqs.78 and 80). Now attention is turned to finding the value of these first round trip amplitude functions by using an auxiliary function $q(x)$ (Ref6:232).

Auxiliary Function $q(x)$. From Eq.(89) a new function, $q_b(x)$, can be introduced which accounts for the slowly varying x-dependence of the diffracted amplitude function $r_1(d,x)$. Let

$$r_1(d,x) = M^{\frac{1}{2}}e^{2Q}e^{2ikD}q_b(x)\left[b-\frac{x}{M}\right]^{-1} \tag{91}$$

Similarly, from Eq.(90) a function, $q_a(x)$, associated with diffracted amplitude $s_1(d,x)$ is introduced:

$$s_1(d,x) = M^{\frac{1}{2}}e^{2Q}e^{2ikD}q_a(x)\left[a+\frac{x}{M}\right]^{-1} \tag{92}$$

Now the $r_n(d,b)$ amplitude function of Eq.(89) can be expressed in terms of $q_b(x)$. From Eq.(78) at $x=b$

$$r_n(d,b) = M^{\frac{n-1}{2}}e^{2(n-1)Q}e^{2ikD(n-1)} \tag{93}$$

$$\times\ r_1(d,k_{bn}(b))$$

where, from Eq.(79)

$$k_{bn}(b) = b\,\frac{M^{-1}+M^{1-n}}{1+M^{-n}} \tag{94}$$

From Eq.(91)

$$r\,(d,k_{bn}(b)) = M^{\frac{1}{2}}e^{2Q}e^{2ikD} \tag{95}$$

$$\times\; q_b\left(b\,\frac{M^{-1}+M^{1-n}}{1+M^{-n}}\right)\left[b\,\frac{1-M^{-2}}{1+M^{-n}}\right]^{-1}$$

Then, substituting Eq.(95) into Eq.(93)

$$r_n(d,b) = M^{\frac{n}{2}}\,e^{2nQ}e^{2nikD}$$

$$\times\; q_b\left(b\,\frac{M^{-1}+M^{1-n}}{1+M^{-n}}\right)\left[b\,\frac{1-M^{-2}}{1+M^{-n}}\right]^{-1} \tag{96}$$

Similarly, using Eq.(80), Eq.(81), and Eq.(92)

$$s_n(d,b) = M^{\frac{n}{2}}\,e^{2nQ}e^{2nikD}$$

$$\times\; q_a\left(a\,\frac{M^{1-n}\left(\frac{b}{a}+M^{-n}\right)-M^{-1}\left(1+\frac{b}{a}M^{-n}\right)}{1-M^{-2n}}\right) \tag{97}$$

$$\times\left[a\,\frac{\left(1+\frac{b}{a}M^{-n}\right)\left(1-M^{-2}\right)}{1-M^{-2n}}\right]^{-1}$$

Now an expression for $q_b(x)$ is needed. Using Eq.(91) and substituting Eq.(96) and Eq.(97) into Eq.(89) gives the following expression for $q_b(x)$

$$q_b(x) = -\left(\frac{id}{2\pi kb^2}\right)^{\frac{1}{2}} \left[\sum_{n=1}^{N} (M^{2n}-1)^{-\frac{1}{2}} M^{\frac{n}{2}} \right.$$

$$\times \exp\left\{\frac{ikb^2}{2d}\,\frac{(1-M^n)^2}{M^{2n}-1}\right\} e^{2nQ_e 2nikD}$$

$$\times\, q_b\left(b\,\frac{(M^{-1}+M^{1-n})(1-M^{-n})}{1-M^{-2n}}\right)\left[b\,\frac{(1-M^{-n})(1-M^{-2})}{1-M^{-2n}}\right]^{-1}$$

$$\times\left[\frac{1-M^{-n}}{1-M^{-2n}} - \frac{\frac{x}{b}\,M^{-1}-M^{-2}}{1-M^{-2}}\right]^{-1}\left(b-\frac{x}{M}\right)$$

$$+\,(M^{2n}-1)M^{\frac{n}{2}} \exp\left\{\frac{ika^2}{2d}\,\frac{\left(\frac{b}{a}+M^n\right)^2}{M^{2n}-1}\right\} \qquad (98)$$

$$\times\, e^{2nQ_e 2nikD} q_a\left(a\,\frac{M^{1-n}\left(\frac{b}{a}+M^{-n}\right)-M^{-1}\left(1+\frac{b}{a}M^{-n}\right)}{1-M^{-2n}}\right)$$

$$\times\left[a\,\frac{\left(1+\frac{b}{a}M^{-n}\right)\left(1-M^{-2}\right)}{1-M^{-2n}}\right]^{-1}$$

$$\times\left[\frac{1+\frac{a}{b}M^{-n}}{1-M^{-2n}} - \frac{\frac{x}{b}\,M^{-1}-M^{-2}}{1-M^{-2}}\right]^{-1}\left(b-\frac{x}{M}\right)$$

$$\left. +\,\hat{f}(d,b)d^{\frac{1}{2}}\left[1-\frac{\frac{x}{b}\,M^{-1}-M^{-2}}{1-M^{-2}}\right]^{-1}\left(b-\frac{x}{M}\right)\right]$$

Upon algebraically simplifying the bracketed terms and

approximating $M^n/(M^{2n}-1)^{\frac{1}{2}}$ by $M^{-n/2}$, the $q_b(x)$ expression of Eq.(98) becomes

$$q_b(x) = -\left(\frac{id}{2\pi kb^2}\right)^{\frac{1}{2}}\left[\sum_{n=1}^{N} M^{\frac{-n}{2}} e^{2nQ} e^{2nikD}\right.$$

$$\times \exp\left\{\frac{ikb^2}{2d}\,\frac{(1-M^n)^2}{M^{2n}-1}\right\} q_b\left(b\,\frac{(M^{-1}+M^{1-n})(1-M^{-n})}{1-M^{-2n}}\right)$$

$$\times(1-M^{-2n})\left[(1-M^{-n}) - \frac{bM^{-n}(1-M^{-2})(1-M^{-n})^2}{(1-M^{-2n})\,(b-\frac{x}{M})}\right]^{-1}$$

$$+M^{\frac{-n}{2}} e^{2nQ} e^{2nikD} \exp\left\{\frac{ika^2}{2d}\,\frac{\left(\frac{b}{a}+M^n\right)^2}{M^{2n}-1}\right\} \tag{99}$$

$$\times\, q_a\left(b\,\frac{M^{1-n}\left(1+\frac{a}{b}M^{-n}\right)-M^{-1}\left(\frac{a}{b}+M^{-n}\right)}{1-M^{-2n}}\right)$$

$$\times\,(1-M^{-2n})\left[\left(\frac{a}{b}+M^{-n}\right)+\frac{bM^{-n}(1-M^{-2})\left(\frac{a}{b}+M^{-n}\right)^2}{(1-M^{-2n})\,(b-\frac{x}{M})}\right]^{-1}$$

$$\left. +\hat{f}(d,b)d^{\frac{1}{2}}b(1-M^{-2})\right]$$

Finally, a similar procedure leads to an expression for $q_a(x)$:

$$q_a(x) = -\left(\frac{id}{2\pi ka^2}\right)^{\frac{1}{2}}\left[\sum_{n=1}^{N} M^{\frac{-n}{2}} e^{2nQ} e^{2nikD}\right.$$

$$\times \exp\left\{\frac{ikb^2}{2d}\,\frac{\left(\frac{a}{b}+M^n\right)^2}{M^{2n}-1}\right\}$$

$$X\ q_b\left(-a\ \frac{M^{1-n}\left(1+\frac{b}{a}M^{-n}\right)-M^{-1}\left(\frac{b}{a}+M^{-n}\right)}{1-M^{-2n}}\right)$$

$$X\ (1-M^{-2n})\left[\left(\frac{b}{a}+M^{-n}\right)+\frac{aM^{-n}(1-M^{-2})\left(\frac{b}{a}+M^{-n}\right)^2}{(1-M^{-2n})\ (a+\frac{x}{M})}\right]$$

$$+M^{\frac{-n}{2}}\ e^{2nQ}e^{2nikD}\ \exp\left\{\frac{ika^2}{2d}\ \frac{(1-M^n)^2}{M^{2n}-1}\right\} \tag{100}$$

$$X\ q_a\left(-a\ \frac{(1-M^{-n})(M^{-1}+M^{1-n})}{1-M^{-2n}}\right)$$

$$X\ (1-M^{-2n})\left[(1-M^{-n})-\frac{aM^{-n}(1-M^{-2})(1-M^{-n})^2}{(1-M^{-2n})\ (a+\frac{x}{M})}\right]^{-1}$$

$$\left.+\hat{f}(d,-a)d^{\frac{1}{2}}a(1-M^{-2})\right]$$

The analysis to this point has expressed the r_n and s_n diffraction amplitude functions of Eq.(26), at the feedback/output mirror plane, in terms of the first round trip diffraction amplitude functions r_1 and s_1. These first round trip amplitudes have then been expressed in terms of the auxiliary functions $q_b(x)$ and $q_a(x)$. Using the relationships between the various diffraction amplitudes on succeeding trip through the resonator to the first round trip amplitudes, the $q(x)$ functions can be expressed in terms of a series expansion in terms of $q(x)$ at distinct

locations on the feedback mirror and the geometrical component, $\hat{f}$, of the field amplitude from Eq.(26). Attention is now turned to expressing this geometrical amplitude in terms of the $q(x)$ functions.

Geometrical Amplitude Functions

Expressions for the geometrical amplitude functions $\hat{f}(d,b)$ and $\hat{f}(d,-a)$ from Eq.(89) and Eq.(90) are needed to evaluate the field amplitude. From the boundary condition Eq.(37) the geometrical terms can be propagated to the feedback mirror along the geometrical rays.

From Eq.(37)

$$\hat{f}(Md,Mx) = (Md)^{-\frac{1}{2}}\hat{g}(Md,Mx) \tag{101}$$

The $\hat{f}$ wave propagates along a ray parallel to the optical axis. The $\hat{f}(Md,Mx)$ amplitude is multiplied by the steady state gain in propagating across the cavity to the feedback mirror. Then

$$\hat{f}(d,Mx) = e^{Q}\hat{f}(Md,Mx) \tag{102}$$

The $\hat{g}(Md,Mx)$ amplitude of Eq.(101) must be propagated backwards to its origin on the feedback mirror. This ray originates at the focal point and intersects the feedback mirror at (d,x) . The steady state gain must be included in this propagation expression also:

$$\hat{g}(Md,Mx) = e^{Q}\hat{g}(d,x) \tag{103}$$

Then from Eq.(102) and Eq.(103)

$$\hat{f}(d,Mx) = M^{-\frac{1}{2}} e^{2Q} d^{-\frac{1}{2}} \hat{g}(d,x) \tag{104}$$

Recalling the boundary condition Eq.(46) at the feedback mirror, this expression can be written as

$$\begin{aligned} \hat{f}(d,Mx) = {} & M^{-\frac{1}{2}} e^{2Q} e^{2ikD} \\ & \times \Bigg[(\rho_N - d)^{-\frac{1}{2}} r_N(d,x) \exp\left\{\frac{ikb^2}{2d}\right\} \\ & + (\rho_N - d)^{-\frac{1}{2}} s_N(d,x) \exp\left\{\frac{ika^2}{2d}\right\} \\ & + \hat{f}(d,x) \Bigg] \end{aligned} \tag{105}$$

From this expression for r_N and s_N in terms of $q_b(x)$ and $q_a(x)$ are needed. From Eq.(78)

$$\begin{aligned} r_N(d,x) = {} & M^{\frac{n-1}{2}} e^{2(N-1)Q} e^{2ikD(N-1)} \\ & \times r_1(d, k_{bn}(x)) \end{aligned} \tag{106}$$

where, for N large

$$k_{bN}(x) = \frac{b}{M} \tag{107}$$

From Eq.(91)

$$r_1(d,k_{bN}(x)) = M^{\frac{1}{2}} e^{2Q} e^{2ikD}$$

$$\times\ q_b\left(\frac{b}{M}\right) \left[b(1-M^{-2})\right]^{-1} \tag{108}$$

Then, combining Eq.(108) and Eq.(106)

$$r_N(d,x) = M^{\frac{N}{2}}\ e^{2NQ} e^{2NikD}$$

$$\times\ q_b\left(\frac{b}{M}\right) \left[b(1-M^{-2})\right]^{-1} \tag{109}$$

Similarly, the expression for the s_n term is found to be

$$s_N(d,x) = M^{\frac{n}{2}}\ e^{2NQ} e^{2NikD}$$

$$\times\ q_a\left(\frac{-a}{M}\right) \left[a(1-M^{-2})\right]^{-1} \tag{110}$$

Then, approximating $M^N/(M^{2N}-1)^{\frac{1}{2}}$ by $M^{-N/2}$, using μ from Eq.(22), and the right hand sides of Eq.(109) and Eq.(110) in Eq.(105) yields

$$\hat{f}(d,Mx) = \mu\left\{d^{-\frac{1}{2}}\mu^N \right.$$

$$\times \left[q_b\left(\frac{b}{M}\right) \exp\left\{\frac{ikb^2}{2d}\right\}\left[b(1-M^{-2})\right]^{-1}\right.$$

$$\times\ q_a\left(\frac{-a}{M}\right) \exp\left\{\frac{ika^2}{2d}\right\} \left[a(1-M^{-2})\right]^{-1}$$

$$\left. + \hat{f}(d,x)\right\} \tag{111}$$

The geometrical expression, $\hat{f}(\rho,x)$, is the amplitude of a plane wave moving toward the right. In the purely geometrical case where the left mirror is "sufficiently large" to reflect all the incident wave, this amplitude is constant across the wave-front. Then

$$\hat{f}(d,Mx) = \hat{f}(d,x) \tag{112}$$

From this, Eq.(111) yields

$$\hat{f}(d,x) = d^{-\frac{1}{2}} \frac{\mu^{N+1}}{1-\mu}$$
$$\times \left[q_b\left(\frac{b}{M}\right)\exp\left\{\frac{ikb^2}{2d}\right\} \left[b(1-M^{-2})\right]^{-1} \right. \tag{113}$$
$$\left. +q_a\left(\frac{-a}{M}\right)\exp\left\{\frac{ika^2}{2d}\right\} \left[a(1-M^{-2})\right]^{-1} \right]$$

Noting that this expression has no x-dependence, as expected, it then follows that

$$\hat{f}(d,b) = \hat{f}(d,-a) = \hat{f}(d,x) \tag{114}$$

Auxiliary Functions in Polynomial Equations

The $q_b(x)$ and $q_a(x)$ auxiliary functions of the $r_1(d,x)$ and $s_1(d,x)$ amplitude functions can now be written in terms of a polynomial expression for the eigenvalues. Using Eq.(113), Eq.(114), Eq.(2), Eq.(3), and Eq.(22) in Eq.(99) gives

$$
\begin{aligned}
q_b(x) = & -\frac{1}{2\pi}\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}} \\
& \times \left\{ \sum_{n=1}^{N} \exp\left\{2\pi i F_{eb}\,\frac{(1-M^{n})^{2}}{M^{2n}-1}\right\} \mu^{n} \right. \\
& \times q_b\left(b\,\frac{(M^{-1}+M^{1-n})(1-M^{-n})}{1-M^{-2n}}\right) \\
& \times (1-M^{-2n})\left[(1-M^{-n}) - \frac{bM^{-n}(1-M^{-2})(1-M^{-n})^{2}}{(1-M^{-2n})\;\left(b-\frac{x}{M}\right)}\right]^{-1} \\
& + \exp\left\{2\pi i F_{ea}\,\frac{\left(\frac{b}{a}+M^{n}\right)^{2}}{M^{2n}-1}\right\} \mu^{n} \qquad\qquad (115) \\
& \times q_a\left(-a\,\frac{M^{-1}\left(1+\frac{b}{a}M^{-n}\right) - M^{1-n}\left(\frac{b}{a}+M^{-n}\right)}{1-M^{-2n}}\right) \\
& \times (1-M^{-2n})\left[\left(\frac{a}{b}+M^{-n}\right) + \frac{bM^{-n}(1-M^{-2})\left(\frac{a}{b}+M^{-n}\right)^{2}}{(1-M^{-2n})\;\left(b-\frac{x}{M}\right)}\right]^{-1} \\
& + \frac{\mu^{N+1}}{1-\mu}\left[q_b\left(\frac{b}{M}\right)\exp\left|2\pi i F_{eb}\right| \right. \\
& \left.\left. + \frac{b}{a}\,q_a\left(\frac{-a}{M}\right)\exp\left|2\pi i F_{ea}\right|\right]\right\}
\end{aligned}
$$

Similarly, for $q_a(x)$

$$
\begin{aligned}
q_a(x) = & -\frac{1}{2\pi}\left(\frac{i}{2F_{ea}}\right)^{\frac{1}{2}} \\
& \times \left\{ \sum_{n=1}^{N} \exp\left\{2\pi i F_{eb} \frac{\left(\frac{a}{b}+M^{n}\right)^2}{M^{2n}-1}\right\} \mu^n \right. \\
& \times q_b\left(b\,\frac{M^{-1}\left(1+\frac{a}{b}M^{-n}\right) - M^{1-n}\left(\frac{a}{b}+M^{-n}\right)}{1-M^{-2n}}\right) \\
& \times (1-M^{-2n})\left[\left(\frac{b}{a}+M^{-n}\right) + \frac{aM^{-n}(1-M^{-2})\left(\frac{b}{a}+M^{-n}\right)^2}{(1-M^{-2n})\ \ (a+\frac{x}{M})}\right]^{-1} \\
& + \exp\left\{2\pi i F_{ea} \frac{(1-M^{n})^2}{M^{2n}-1}\right\} \mu^n \qquad (116) \\
& \times q_a\left(-a\,\frac{(1-M^{-n})(M^{-1}+M^{1-n})}{1-M^{-2n}}\right) \\
& \times (1-M^{-2n})\left[(1-M^{-n}) - \frac{aM^{-n}(1-M^{-2})(1-M^{-n})^2}{(1-M^{-2n})\ (a+\frac{x}{M})}\right]^{-1} \\
& + \frac{\mu^{N+1}}{1-\mu}\left[\frac{a}{b}\, q_b\left(\frac{b}{M}\right) \exp\left|2\pi i F_{eb}\right| \right. \\
& \left.\left. + q_a\left(-\frac{a}{M}\right) \exp\left|\, 2\pi i F_{ea}\right| \right] \right\}
\end{aligned}
$$

In summary, the r_n and s_n diffraction amplitude functions of Eq.(26) were related to the amplitudes r_1 and s_1. These first round trip diffraction amplitude functions were related to the auxiliary functions $q_b(x)$ and $q_a(x)$. The geometrical amplitude function, $\hat{f}$, from Eq.(26) was also related to the $q_b(x)$ and $q_a(x)$ functions. Equations (115)and (116) are the resulting expressions for the auxiliary functions. The next chapter, using several assumptions, develops the eigenvalue polynomial from these equations. Each eigenvalue can then be used to solve for the r_n, s_n, and $\hat{f}$ of Eq.(26) for each mode.

V. Eigenvalue Equations and Limitations

The purpose of this chapter is the development of expressions from which the eigenvalues of the off-center unstable resonator can be determined and their limitations. The first section develops a polynomial equation using an approximation on the $q(x)$ functions. The eigenvalues are the roots of this equation. The second section discussed the validity of the approximations made in the first section. It then proposes a matrix equation which can be used to determine the eigenvalues and the behavior of the $q(x)$ function. The final section discusses the limitations on the application of the eigenvalue expressions.

Polynomial Expression Assuming Constant $q(x)$

Development of Polynomial. To obtain a tractable polynomial equation for the eigenvalues, μ , of the centered resonator, Moore and McCarthy (Ref 6:233) make two approximations. First, the function $q(x)$ is assumed to be a constant. Secondly, with the exception of exponential terms, M^{-n} is neglected when compared to one. To obtain the equivalent expressions for the off-center case requires the approximations

$$q_b(x) = q_b \tag{117}$$

$$q_a(x) = q_a \tag{118}$$

$$M^{-n} << \text{minimum of } \left| 1, \frac{a}{b}, \text{or } \frac{b}{a} \right| \tag{119}$$

Here, q_b and q_a are constants. In the case of the centered resonator $q_a = q_b$. Equation (119) does not apply for arguments of exponential terms. Using these approximations in Eq.(115), and dividing through by q_b yields

$$\begin{aligned}
1 = &- \frac{1}{2\pi} \left(\frac{i}{2F_{eb}} \right)^{\frac{1}{2}} \\
&X \left\{ \sum_{n=1}^{N} \mu^n \left[\exp \left\{ 2\pi i F_{eb} \frac{(1-M^n)^2}{M^{2n}-1} \right\} \right. \right. \\
&\left. + \frac{q_a}{q_b} \frac{b}{a} \exp \left\{ 2\pi i F_{ea} \frac{\left(\frac{b}{a} + M^n\right)^2}{M^{2n}-1} \right\} \right] \\
&+ \frac{\mu^{N+1}}{1-\mu} \left[\exp \left\{ 2\pi i F_{eb} \right\} \right. \\
&\left. \left. + \frac{q_a}{q_b} \frac{b}{a} \exp \left\{ 2\pi i F_{ea} \right\} \right] \right\}
\end{aligned} \tag{120}$$

Similarly, after dividing through by q_a rather than q_b, Eq.(116) gives

$$1 = -\frac{1}{2\pi}\left(\frac{i}{2F_{ea}}\right)^{\frac{1}{2}}$$

$$\times \left\{ \sum_{n=1}^{N} \mu^{n} \left[\frac{q_b}{q_a}\,\frac{a}{b}\, \exp\left\{ 2\pi i F_{eb}\, \frac{\left(\frac{a}{b} + M^{n}\right)^{2}}{M^{2n}-1} \right\} \right. \right.$$

$$\left. + \exp\left\{ 2\pi i F_{ea}\, \frac{(1-M^{n})^{2}}{M^{2n}-1} \right\} \right] \qquad (121)$$

$$+ \frac{\mu^{N+1}}{1-\mu} \left[\frac{q_b}{q_a}\,\frac{a}{b}\, \exp\left\{ 2\pi i F_{ea} \right\} \right.$$

$$\left. \left. + \exp\left\{ 2\pi i F_{ea} \right\} \right] \right\}$$

The laser parameters (wavelength, magnification, mirror separation, and the distances a and b of the feedback mirror edges from the optical axis) are assumed to be known. Then Eq.(120) and Eq.(121) have two unknown parameters, μ and (q_a/q_b), which can be determined. After multiplying through Eq.(121) by (bq_a/aq_b) and subtracting from Eq.(120), the resulting equation can be solved for (q_a/q_b), resulting in the following expression

$$\frac{q_a}{q_b} = \left\{ 1 - \frac{1}{2\pi} \left[\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}} - \left(\frac{i}{2F_{ea}}\right)^{\frac{1}{2}} \right] \right.$$

$$X \sum_{n=1}^{N} \mu^n \left[\exp\left\{ 2\pi i F_{eb} \frac{(1-M^n)^2}{M^{2n}-1} \right\} \right.$$

$$\left. \left. - \exp\left\{ 2\pi i F_{eb} \frac{\left(\frac{a}{b} + M^n\right)^2}{M^{2n}-1} \right\} \right] \right\} \tag{122}$$

$$\div \left\{ \frac{b}{a} \left(1 + \frac{1}{2\pi} \left[\left(\frac{i}{2F_{eb}} \right)^{\frac{1}{2}} - \left(\frac{i}{2F_{ea}} \right)^{\frac{1}{2}} \right] \right. \right.$$

$$X \sum_{n=1}^{N} \mu^n \left[\exp\left\{ 2\pi i F_{ea} \frac{\left(\frac{b}{a} + M^n\right)^2}{M^{2n}-1} \right\} \right.$$

$$\left. \left. \left. - \exp\left\{ 2\pi i F_{ea} \frac{(1-M^n)^2}{M^{2n}-1} \right\} \right] \right) \right\}$$

The eigenvalue polynomial in terms of μ can then be obtained by substituting this expression into Eq.(120) for q_a/q_b. Bringing all terms on the left to the right hand side of the equation, multiplying through by $(1-\mu)$, and collecting like summations yields the eigenvalue polynomial:

$$0 = \sum_{n=1}^{N} \sum_{k=1}^{N} \mu^n \mu^k (1-\mu) A_{nk}$$

$$+ \sum_{n=1}^{N} \mu^n (B_n (1-\mu) + \mu^{N+1} C_n) \tag{123}$$

$$+ \mu^{N+1} D + \mu - 1$$

where

For A_{nk}

$$A_{nk} = -\frac{1}{2\pi}\ \frac{i}{2F_{eb}}^{\frac{1}{2}} \left[-\frac{1}{2\pi}\left[\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}} - \left(\frac{i}{2F_{ea}}\right)^{\frac{1}{2}}\right]\right]$$

$$X \left\{ \exp\left\{ 2\pi i F_{eb} \frac{(1-M^{n})^2}{M^{2n}-1} \right\} \right.$$

$$X \left[\exp\left\{ 2\pi i F_{ea} \frac{\left(\frac{b}{a} + M^{k}\right)^2}{M^{2k}-1} \right\} \right.$$

$$\left. -\exp\left\{ 2\pi i F_{ea} \frac{(1-M^{k})^2}{M^{2k}-1} \right\} \right] \qquad (124)$$

$$-\exp\left\{ 2\pi i F_{ea} \frac{\left(\frac{b}{a} + M^{n}\right)^2}{M^{2n}-1} \right\}$$

$$X \left[\exp\left\{ 2\pi i F_{eb} \frac{(1-M^{k})^2}{M^{2k}-1} \right\} \right.$$

$$\left. \left. -\exp\left\{ 2\pi i F_{eb} \frac{\left(\frac{a}{b} + M^{k}\right)^2}{M^{2k}-1} \right\} \right] \right\}$$

For B_n

$$B_n = -\frac{1}{2\pi} \left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}}$$

$$X \left[\exp\left\{ 2\pi i F_{eb} \frac{(1-M^{n})^2}{M^{2n}-1} \right\} \right.$$

$$\left. + \exp\left\{ 2\pi i F_{ea} \frac{\left(\frac{b}{a} + M^{n}\right)^2}{M^{2n}-1} \right\} \right] \qquad (125)$$

$$+ \frac{1}{2\pi}\left[\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}} - \left(\frac{i}{2F_{ea}}\right)^{\frac{1}{2}}\right]$$

$$X\left[\exp\left\{2\pi i F_{ea}\frac{\left(\frac{b}{a}+M^{n}\right)^{2}}{M^{2n}-1}\right\}\right.$$

$$\left. - \exp\left\{2\pi i F_{ea}\frac{(1-M^{n})^{2}}{M^{2n}-1}\right\}\right]$$

for C_n $$C_n = -\frac{1}{2\pi}\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}}$$

$$X - \frac{1}{2\pi}\left[\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}} - \left(\frac{i}{2F_{ea}}\right)^{\frac{1}{2}}\right]$$

$$X\left(\exp\left\{2\pi i F_{eb}\right\}\left[\exp\left\{2\pi i F_{ea}\frac{\left(\frac{b}{a}+M^{n}\right)^{2}}{(M^{2n}-1)}\right\}\right.\right.$$

$$\left. - \exp\left\{2\pi i F_{ea}\frac{(1-M^{n})^{2}}{M^{2n}-1}\right\}\right] \qquad (126)$$

$$-\exp\left\{2\pi i F_{ea}\right\}\left[\exp\left\{2\pi i F_{eb}\frac{(1-M^{n})^{2}}{M^{2n}-1}\right\}\right.$$

$$\left.\left. -\exp\left\{2\pi i F_{eb}\frac{\left(\frac{a}{b}+M^{n}\right)^{2}}{M^{2n}-1}\right\}\right]\right)$$

and for D_n

$$D = -\frac{1}{2\pi}\left(\frac{i}{2F_{eb}}\right)^{\frac{1}{2}}$$

$$X\left[\exp\left\{2\pi i F_{ea}\right\} + \exp\left\{2\pi i F_{eb}\right\}\right] \qquad (127)$$

The eigenvalues for the modes of the off-center unstable confocal resonator are the roots of the polynomial in Eq.(123). In the case of the centered resonator where $a=b$, these expressions reduce to Moore and McCarthy's (Ref 6:233).

<u>Field on the Feedback Mirror</u>. The right-traveling field on the feedback mirror can now be determined to within a constant multiplicative factor, q_a^{-1} or q_b^{-1}. From Eq.(91) and Eq.(92), using Eq.(22), $r_1(d,x)$ and $s_1(d,x)$ can be expressed as

$$r_1(d,x) = M\mu q_b \left[b - \frac{x}{M}\right]^{-1} \tag{128}$$

$$s_1(d,x) = M\mu q_a \left[a + \frac{x}{M}\right]^{-1} \tag{129}$$

Substituting these into Eq.(78) and Eq.(79) yields

$$r_n(d,x) = M^n \mu^n q_b \left[b - \frac{k_{bn}(x)}{M}\right]^{-1} \tag{130}$$

$$s_n(d,x) = M^n \mu^n q_a \left[a + \frac{k_{an}(x)}{M}\right]^{-1} \tag{131}$$

Using Eq.(130) and Eq.(131) in Eq.(26) and also approximating $M^{2n}/(M^{2n}-1)^{\frac{1}{2}}$ as unity, the right-traveling field amplitude, to within the multiplicative constant q_b^{-1}, is given by

$$f(d,x)q_b^{-1} = \sum_{n=1}^{N} d^{-\frac{1}{2}} \mu^n$$

$$\times \left(\exp\left\{ 2\pi i F_{eb} \frac{\left(\frac{x}{b} - M^n\right)^2}{M^{2n}-1} \right\} \left[b - \frac{k_{bn}(x)}{M} \right]^{-1} \right.$$

$$\left. + \frac{q_a}{q_b} \exp\left\{ 2\pi i F_{ea} \frac{\left(\frac{x}{a} + M^n\right)^2}{M^{2n}-1} \right\} \left[a + \frac{k_{an}(x)}{M} \right]^{-1} \right)$$

$$+ d^{-\frac{1}{2}} \frac{\mu^{N+1}}{1-\mu} \left(\exp\left\{ 2\pi i F_{eb} \right\} \left[b(1-M^{-2}) \right]^{-1} \right. \tag{132}$$

$$\left. + \frac{q_a}{q_b} \exp\left\{ 2\pi i F_{ea} \right\} \left[a(1-M^{-2}) \right]^{-1} \right)$$

<u>Validity of Approximations and Proposed Solution</u>

For the centered resonator case the feedback mirror edges are an equal distance from the optical axis. Then $a=b$ and Eq.(115) becomes

$$q(x) = -\frac{1}{2\pi} \left(\frac{i}{2F_e}\right)^{\frac{1}{2}}$$

$$\times \left\{ \sum_{n=1}^{N} \mu^n \exp\left\{ 2\pi i F_e \frac{M^n-1}{1+M^n} \right\} \right.$$

$$\times\, q\left(a\, \frac{M^{-1}+M^{1-n}}{1+M^{-n}} \right) \left(1+M^{-n}\right)$$

$$\times \left[1 - \frac{aM^{-n}(1-M^{-2})}{(1+M^{-n})(a-\frac{x}{M})} \right]^{-1} \tag{133}$$

$$+ \exp\left\{2\pi i F_e \frac{1+M^n}{M^n-1}\right\} q_a\left(-a\,\frac{M^{-1}-M^{1-n}}{1-M^{-n}}\right)$$

$$X\ (1-M^{-n})\left[1+\frac{aM^{-n}(1-M^{-2})}{(1-M^{-n})(a-\frac{x}{M})}\right]^{-1}$$

$$\left. + \frac{\mu^{N+1}}{1-\mu}\left[q\left(\frac{a}{M}\right)+q\left(\frac{-a}{M}\right)\right]\exp\left\{2\pi i F_e\right\}\right\}$$

This expression reduces to Moore and McCarthy's (Ref 6:232) expression only if $q(x) = q(-x)$ for the case of symmetric modes in the centered geometry.

The approximation that M^{-n} can be neglected when compared to one is valid for large M, and n greater than one or two. However, for cases where the magnification approaches unity it can easily be seen that neglecting the M^{-n} terms (for small n) when comparing this term to one is a poor approximation.

In Moore and McCarthy's analysis for the centered resonator geometry the auxiliary function, $q(x)$, is assumed to be a constant. This allows the q-terms to be divided out of Eq.(133) to obtain the eigenvalue polynomial. It is not clear whether or not this is a valid approximation. Moore and McCarthy offer no justification for it. A method of solving for the eigenvalues and also examining the behavior of the $q(x)$ function without neglecting M^{-n} terms can be developed however.

Equation (133) expresses $q(x)$ in terms of a summation over $q(x_k)$ values where x_k are specific locations on the output mirror. Each $q(x_k)$ can be expressed in the form of Eq.(133). For any upper bound, N, of the summation this leads to a system of $2N+1$ equations of the form given in Eq.(133). This system of equations can be written as a non-linear matrix eigenvalue problem. If this can be solved for the true eigenvalues (ie. no constant $q(x)$ assumption made) then the resulting eigenvectors can be used to determine the behavior of the $q(x)$ function for various modes in the centered resonator geometry.

To obtain the matrix equation form the following symbolism is introduced:

$$C = -\frac{1}{2\pi}\left(\frac{i}{2F_e}\right)^{\frac{1}{2}} \tag{134}$$

$$E_{\pm n} = \exp\left\{2\pi i F_e \frac{M^n \mp 1}{M^n \pm 1}\right\} \tag{135}$$

$$E_0 = \exp\left\{2\pi i F_e\right\} \tag{136}$$

$$x_{\pm k} = a\,\frac{M^{-1} \pm M^{1-k}}{1 \pm M^{-k}} \tag{137}$$

$$x_0 = \frac{a}{M} \tag{138}$$

$$A_{\pm n}(x_{\pm k}) = (1\pm M^{-n})\left[1 \mp \frac{aM^{-n}(1-M^{-2})}{(1\pm M^{-n})\left(a-\frac{x_{\pm k}}{M}\right)}\right]^{-1} \tag{139}$$

where $n=1,2,\ldots,N$ and $k=0,1,\ldots N$.

Replacing $q(x)$ by $q(x_k)$ in the left hand side of Eq. (133) and using the above symbolism leads to an equation for each $q(x_k)$ of the form

$$\begin{aligned} 0 = \sum_{n=1}^{N} & CE_{+n}A_{+n}(x_k)q(x_{+n})\mu^n \\ & +CE_{-n}A_{-n}(x_k)q(x_{-n})\mu^n \\ & +\frac{\mu^{N+1}}{1-\mu}\, 2CE_0 q(x_0) \\ & -q(x_k) \end{aligned} \tag{140}$$

The $2N+1$ equations of this form (from the $2N+1$ values of k) can then be written in the homogeneous matrix equation form

$$\underline{A}(\mu)\cdot\underline{q}=0 \tag{141}$$

where $\underline{A}(\mu)$ is a square matrix of dimension $(2N+1)$ and $\underline{q}$ a vector of $(2N+1)$ elements. The matrix elements of $\underline{A}(\mu)$ consist of the coefficients of $q(x_n)$ in Eq.(140). Appendix E develops representative elements in this matrix. These

coefficients contain only one unknown parameter, μ, the eigenvalue. This is then a non-linear matrix eigenvalue problem. Problems of this type are very difficult to solve in general. If this one can be solved it will yield the true eigenvalues, μ, and the elements of the eigenvector $\underline{q}$, for each eigenvalue can be used in the right-hand side of Eq.(133) to determine the behavior of the function $q(x)$.

Limitations

There are limitations to the range of applicability of these expressions which are due to the geometry of the resonator and approximations used in this development.

The polynomial expression, used to determine the eigenvalues for the modes, is valid only for the positive branch unstable confocal geometry with an off-center (or centered in the case where $a=b$) feedback mirror. Since a magnification of unity cannot be achieved by this configuration, the expressions developed become increasingly inaccurate as M approaches one. At M equal to one the expressions blow up as illustrated by the geometrical contribution to the field in Eq.(132).

The asymptotic expansion used in this development has a singularity at the geometrical shadow boundary. The eigenvalue expressions are correct but the field can only be determined well within the shadow boundaries.

As an effective Fresnel number for the resonator approaches unity, this analysis becomes increasingly inaccurate. This

stems from the asymptotic expansion where terms of higher order than F_{ea}^{-1} are neglected. As the effective Fresnel number approaches unity, more terms must be included in the series expansion for the diffraction integral. For the off-center geometry this requires that the optical axis not pass very close to an edge of the feedback mirror in this analysis.

Another limitation to this analysis is due to the Fresnel approximation used in the phase terms of the diffraction amplitude functions. When these amplitudes are used in the diffraction integral the resulting field amplitude becomes valid a small distance from the feedback mirror. How close this distance is to the mirror is limited by the Fresnel approximation. This implies that the left-traveling field at a point close to the feedback mirror consists of the contributions at the point where the geometric ray intersects the feedback mirror and from points in some small region about that point that satisfy the Fresnel approximation. Contributions from other points outside this region are thus assumed to be negligible.

VI. Conclusions and Recommendations

The primary objective of this work has been accomplished. An eigenvalue polynomial for the modes of an off-center unstable confocal resonator has been developed. In the special case of the centered feedback mirror, the off-center eigenvalue polynomial should reduce to the form developed by Moore and McCarthy(Ref 6:233). This has been verified, validating the off-center eigenvalue polynomial in this limit.

In fulfillment of the secondary objective, a non-linear matrix eigenvalue problem has been developed to examine the behavior of the $q(x)$ function for the centered case. If solutions to this problem are feasible, the approximation of $q(x)$ as a constant can be examined.

It is recommended that methods of solving the $q(x)$ non-linear matrix eigenvalue problem be investigated. If solving the problem proves feasible, the behavior of the $q(x)$ function should be investigated. If $q(x)$ varies significantly from a constant value, its effects on the modes of the resonator should be determined.

If $q(x)$ can be evaluated for the centered case, it is recommended that this technique be extended to examine the behavior of the functions $q_b(x)$ and $q_a(x)$ for the off-center case. Their combined effect in the off-center case may make modes for this resonator geometry more sensitive to deviations

of the $q(x)$ functions from constant values.

With the eigenvalue polynomial for the off-center geometry, the computer program developed by Moore and McCarthy should be modified to calculate the eigenvalues and the modes for the off-center geometry. A parametric study could then be undertaken of the eigenvalues and modes of various resonator configurations.

This asymptotic analysis should be extended to allow the determination of the field beyond the geometrical shadow boundaries. This could be accomplished using either the method employed by Moore and McCarthy (Ref 6:234) or the uniform asymptotic expansion of Horwitz (Ref 4:172). With the field across the entire output plane the mode intensity distribution could be determined. Then Fast Fourier Transform techniques could be used to calculate far field intensity distributions.

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Appendix A

Phase Effects of Tilt
Versus Rotation of the Feedback Mirror

The purpose of this appendix is to illustrate the differences that can occur in the distance a ray from the feedback mirror travels in an off-center resonator due to tilting the feedback mirror rather than rotating it about its center of curvature. The difference in distance traveled is related to phase differences.

Rotated Feedback Mirror

If the feedback mirror is centered on the optical axis and then rotated about its center of curvature, an off-center resonator of the same geometry as used throughout this report is generated. This rotated off-center unstable confocal resonator is illustrated in Fig.(5).

In this case the (ρ,θ) coordinate system has its origin at the center of curvature of the feedback mirror rather than the focal point. The (z,x) coordinate system has its origin at the intersection of the optical axis and the large mirror. The mirrors are separated by a distance D . The radius of curvature of the feedback mirror is $2d$. Also, the distance separating the centers of curvature of the two mirrors is S . For the confocal case S equals D.

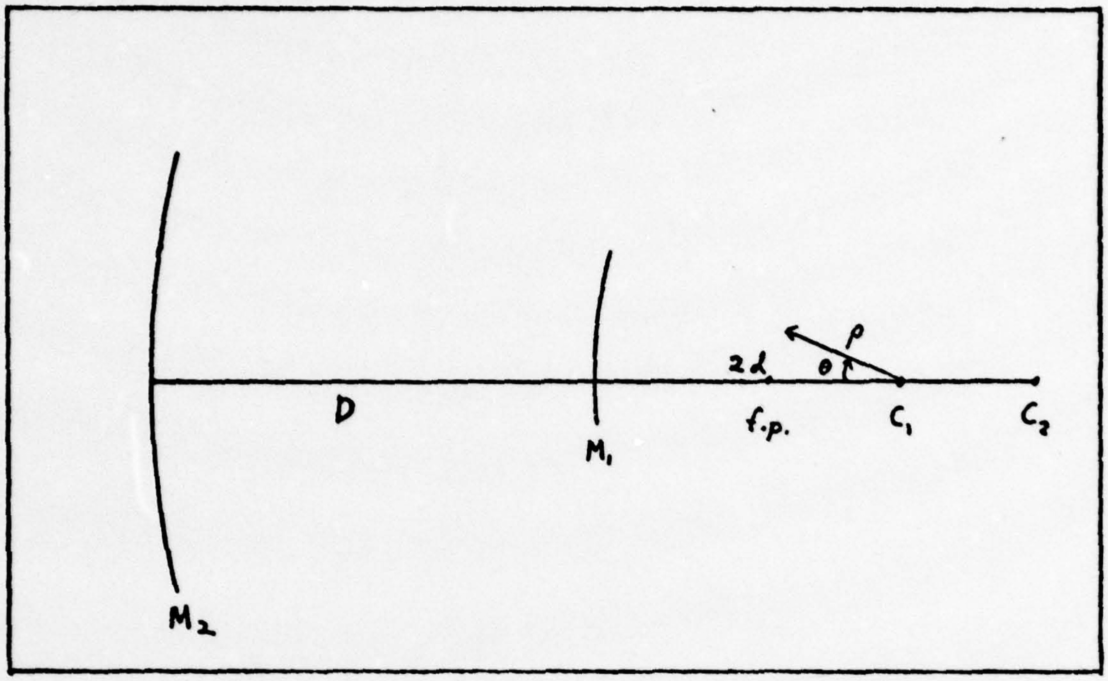

Fig. 5 ROTATED MIRROR OFF-CENTER RESONATOR

For ease of computation, the case where D equals d , i.e. resonators with a magnification of two are considered.

For a point $(2d,\theta)$ on the mirror, the distance, L , from it to the point where the optical axis intersects the large mirror is given by

$$L = (z^2+x^2)^{\frac{1}{2}} \tag{A-1}$$

where

$$z = D+2d(1-\cos\theta)$$

$$x = 2d \sin\theta$$

In the paraxial range of θ, $\sin\theta=\theta$ and $\cos\theta=1$. Then Eq.(A-1) becomes

$$L \doteq (D^2+4d^2\theta^2)^{\frac{1}{2}} \quad \text{(A-2)}$$

In comparison to the centered case the distance is identical for a similar point $(2d,\theta)$. So, no phase changes must be accounted for in extending the analysis to the off-centered confocal resonator when the off-center condition is introduced by a rotation of the feedback mirror about its center of curvature.

<u>Tilted Feedback Mirror</u>

If the centered feedback mirror is now tilted by an angle δ about its center, the optical axis shifts and an off-center resonator is generated. This geometry is depicted in Fig.(6).

This resonator is no longer confocal since S', the new separation of the centers of curvature, and the mirror separation, D', are no longer equal. From the law of cosines, the distance S' is given by

$$S' = \left[4d^2+(2d+S)^2-4d(2d+S)\cos\delta\right]^{\frac{1}{2}} \quad \text{(A-3)}$$

Recalling, that $S=d$

$$S' = d\left[12(1-\cos\delta)+1\right]^{\frac{1}{2}} \quad \text{(A-4)}$$

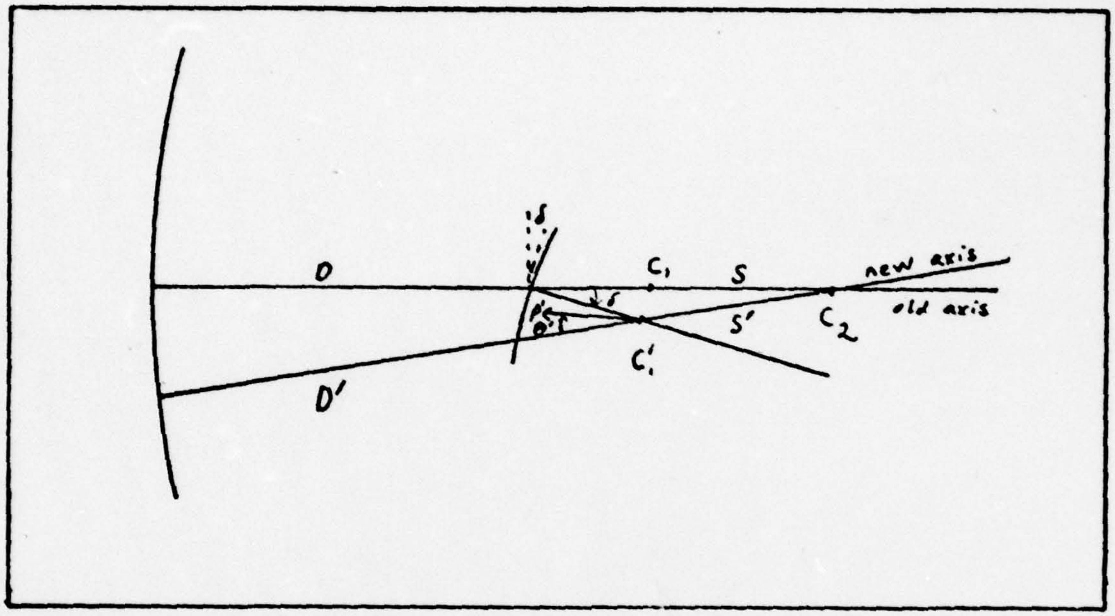

Fig. 6 TILTED MIRROR OFF-CENTER RESONATOR

Then the change in the center of curvature separation (and also in mirror separation) is given by

$$S'-S = \Delta S = \left[12(1-\cos\delta)+1\right]^{\frac{1}{2}}d-d \qquad \text{(A-5)}$$

If it is assumed that δ is small then $(1-\cos\delta)$ is also small. Using the binomial approximation of the bracketed term in Eq.(A-5)

$$\Delta S = 6d(1-\cos\delta) \qquad \text{(A-6)}$$

Using the series expansion of $\cos\delta$

$$\Delta S = 3d\delta^2 \qquad \text{(A-7)}$$

Then the mirror separation, D' , is given by

$$D' = D-\Delta S \tag{A-8}$$

Now the distance, L' , to feedback mirror point $(2d,\theta)$ from the intersection of the optical axis with the large mirror is given by

$$L' = (z'^2+x'^2)^{\frac{1}{2}} \tag{A-9}$$

where

$$z' = D'+2d(1-\cos\theta)$$

$$x' = 2d\sin\theta$$

Again, for small θ , Eq.(A-9) can be written

$$L' = (D'^2+4d^2\theta^2)^{\frac{1}{2}} \tag{A-10}$$

Using Eq.(A-8) and Eq.(A-7)

$$L' = \left[(D-3d\delta^2)^2+4d^2\theta^2\right]^{\frac{1}{2}} \tag{A-11}$$

This becomes

$$L' = \left[D^2+4d^2\theta^2+3d\delta^2(3d\delta^2-2D)\right]^{\frac{1}{2}} \tag{A-12}$$

For small δ , using the binomial approximation, Eq.(A-12) becomes

$$L' = L+\tfrac{1}{2}\Delta S(\Delta S-2D) \tag{A-13}$$

Then

$$\Delta L = \tfrac{1}{2}\Delta S(\Delta S-2D) \tag{A-14}$$

This can be neglected in phase terms if

$$\Delta Lk \ll \frac{\pi}{2} \tag{A-15}$$

where k is the wave number.

Recalling that D equals d in the case being considered, and neglecting terms of order δ^4, it is found that the condition

$$\delta \ll \left(\frac{\pi}{6d^2k}\right)^{\frac{1}{2}} \tag{A-16}$$

is required so that significant phase changes do not occur. For optical frequencies this limits δ to very small values.

Parenthetically, a tilted off-axis resonator is equivalent to an off-center non-confocal resonator (Ref 8:2242). A tilt can then be viewed as a rotation about the center of curvature and a shift of the feedback mirror along the optical axis.

Appendix B

Virtual Images of Feedback Mirror Edges

The field amplitude functions $f(z,x)$ and $g(\rho,\theta)$ in the laser cavity are expanded in terms of geometrical amplitude functions and edge diffraction amplitude functions. The diffraction amplitude functions can be viewed as cylindrical wavelets originating at the virtual images of the feedback mirror edges. The location of the virtual image sources for the cylindrical wavelets can be determined by matrix optics techniques.

Given the geometry of the off-center unstable confocal resonator in Fig. (7), the image of any point (d,x) on the feedback mirror can be found by the matrix equation

$$\begin{bmatrix} x' \\ \theta' \end{bmatrix} = T_2 R T_1 \begin{bmatrix} x \\ \theta \end{bmatrix} \qquad \text{(B-1)}$$

where

x = transverse coordinate of object point

θ = angle of ray leaving object point with respect to optical axis

T_1 = translation matrix from object point to mirror

R_m = reflection matrix for imaging mirror

T_2 = translation matrix from mirror for exit ray

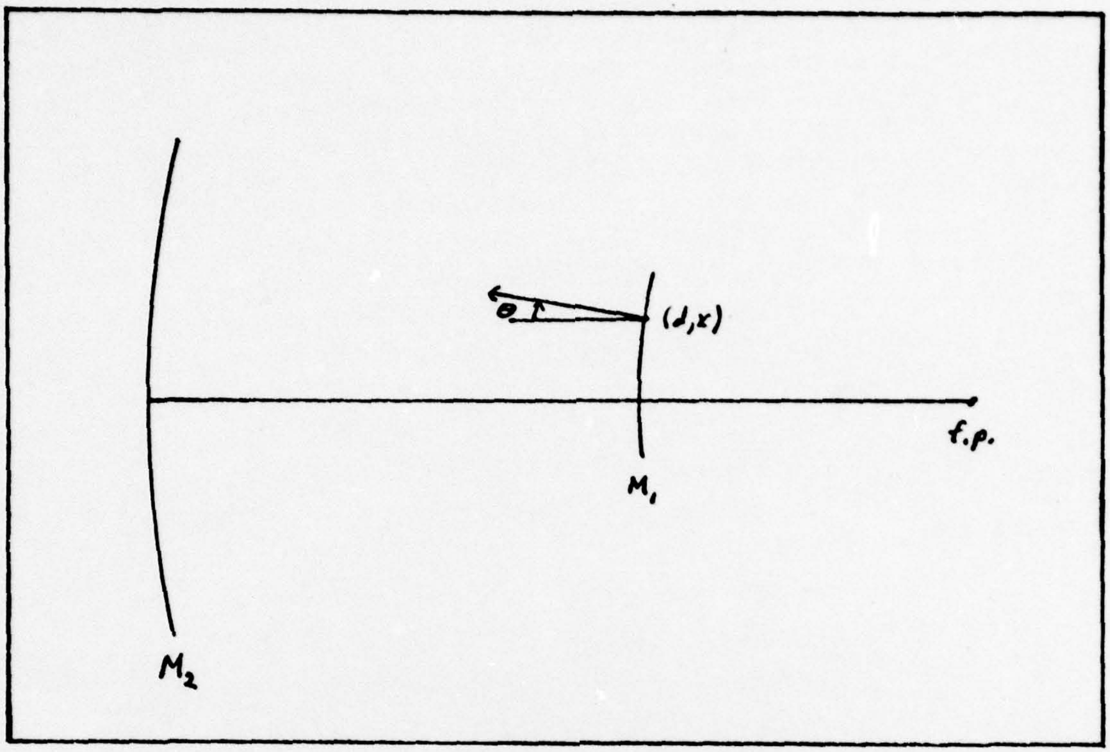

Fig. 7 RAY FROM FEEDBACK MIRROR

x' = transverse coordinate for exit ray

θ' = angle of exit ray with respect to optical axis

Substituting for the translation and reflection matrices (Ref 5:84-100) yields

$$\begin{bmatrix} x' \\ \theta' \end{bmatrix} = \begin{bmatrix} 1 & L' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \qquad \text{(B-2)}$$

where

L = translation distance parallel to optical axis from object point to mirror

L' = translation distance from mirror along optical axis

R = radius of curvature of mirror

The sign convention for the radius of curvature is chosen to be positive for concave mirrors and negative for convex mirrors.

Performing the matrix multiplication of T_1 , R , and T_2 in Eq.(B-2) yields

$$\begin{bmatrix} x' \\ \theta' \end{bmatrix} = \begin{bmatrix} 1-\frac{2L'}{R} & L-\frac{2L'L}{R}+L' \\ -\frac{2}{R} & 1-\frac{2L}{R} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \qquad \text{(B-3)}$$

or

$$\begin{bmatrix} x' \\ \theta' \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \qquad \text{(B-4)}$$

The matrix for an image forming system has the M_{12} element equal to zero.

This implies

$$L - \frac{2L'L}{R} + L' = 0 \qquad \text{(B-5)}$$

Given L and R , L' can be determined. The distance of L' is the position along the optical axis of the image point from the reflecting mirror. For L' negative the image is virtual and located behind the reflecting surface, for L' positive the image is real and in front of the mirror.

For an image forming system the M_{11} matrix element in Eq.(B-4) is the transverse magnification M_T:

$$M_T = 1 - \frac{2L'}{R} \qquad \text{(B-6)}$$

The transverse distance of the image from the optical axis can be determined from

$$x' = M_T x \qquad \text{(B-7)}$$

So the object point (L,x) will be imaged at (L',x') relative to the imaging mirror.

For the off-center unstable confocal resonator geometry of Fig.(7), the feedback mirror M_1 has a radius of curvature

$$R_1 = -2d \qquad \text{(B-8)}$$

The large mirror M has a radius of curvature

$$R_2 = 2(D+d) \qquad \text{(B-9)}$$

The separation between the mirrors is distance D. The feedback mirror is distance d from the focal point.

To locate points relative to the focal point, the paraxial coordinates (ρ,x) are used where positive ρ is the distance to the left of the focal point and positive x is the transverse distance above the optical axis. In this coordinate system, a point on the feedback mirror is (d,x).

Given an object point (ρ_{obj},x_{obj}) the translation distance L_1 from the feedback mirror is

$$L_1 = \rho_{obj} - d \qquad \text{(B-10)}$$

Then using Eq.(B-10) in Eq.(B-5) yields

$$L_1' = -\frac{L_1 d}{L_1+d} \tag{B-11}$$

The ρ coordinate of the image point is then

$$\rho_{im} = d+L_1' \tag{B-12}$$

The transverse magnification of the feedback mirror using Eq.(B-8) in Eq.(B-6) is

$$M_{T_1} = \frac{d+L_1'}{d} = \frac{\rho_{im}}{d} \tag{B-13}$$

Using this in Eq.(B-7) yields

$$x_{im} = \frac{d+L_1'}{d} \; x_{obj} = \frac{\rho_{im}}{d} x_{obj} \tag{B-14}$$

Thus, the image location formed by the feedback mirror is given by

$$(\rho_{im}, x_{im}) = (d+L_1', \; \frac{d+L'}{d} \; x_{obj}) \tag{B-15}$$

Similarly, given the object point (ρ_{obj}, x_{obj}) to be imaged by the large mirror M_2

$$L_2 = D+d-\rho_{obj} = dM-\rho_{obj} \tag{B-16}$$

$$L_2' = \frac{L_2(D+d)}{L_2-(D+d)} = \frac{L_2 dM}{L_2-dM} \tag{B-17}$$

$$\rho_{im} = D+d-L_2' = dM-L_2' \tag{B-18}$$

$$M_{T_2} = \frac{D+d-L_2'}{D+d} = \frac{\rho_{im}}{dM} \tag{B-19}$$

$$x_{im} = \frac{D+d-L_2'}{D+d}\, x_{obj} = \frac{\rho_{im}}{dM}\, x_{obj} \tag{B-20}$$

where

$$M = \frac{D+d}{d} \tag{B-21}$$

The object points on the feedback mirror undergo transverse magnification M given above.

Then, the location of images formed by the large mirror is given by

$$(\rho_{im}, x_{im}) = (D+d-L_2', \frac{D+d-L_2'}{D+d}\, x_{obj}) \tag{B-22}$$

or

$$(\rho_{im}, x_{im}) = (dM-L_2', \frac{dM-L_2'}{dM}) \tag{B-23}$$

Using Eq.(B-16)through Eq.(B-20), the virtual image of a point (d,x) on the feedback mirror is found to be

$$(\rho_1, x_1) = (dM^2, xM) \tag{B-24}$$

The image of this point (ρ_1, x_1) produced by the feedback mirror, using Eq.(B-10) through Eq.(B-15) is located at

$$(\rho_{-1}, x_{-1}) = (dM^{-2}, xM^{-1}) \qquad \text{(B-25)}$$

It can easily be shown that continued reflection of the succeeding virtual images yields the general form

$$(\rho_n, x_n) = (dM^{2n}, xM^n) \ ; \ n=\pm1,\pm2,\ldots,\pm N \qquad \text{(B-26)}$$

where the positive subscript indicates the nth reflection by the large mirror, and the negative subscript is the nth reflection by the feedback mirror.

Given the edges of the feedback mirror located at (d,b) and $(d,-a)$, the virtual images of the edges are given by

$$(\rho_n, b_n) = (dM^{2n}, bM^n) \ ; \ n=\pm1,\pm2,\ldots\pm N \qquad \text{(B-27)}$$

$$(\rho_0, b_0) = (d,b) \qquad \text{(B-28)}$$

and

$$(\rho_n, -a_n) = (dM^{2n}, -aM^n) \ ; \ n=\pm1,\pm2,\ldots\pm N \qquad \text{(B-29)}$$

$$(\rho_0, -a_0) = (d,-a) \qquad \text{(B-30)}$$

It can be seen that these points lie on the halves of two different parabolas centered on the optical axis as shown in Fig. (8). These parabolas have a common point at the focal point of the resonator. This is the (ρ_n, b_n) and $(\rho_n, -a_n)$ point for $n \to -\infty$.

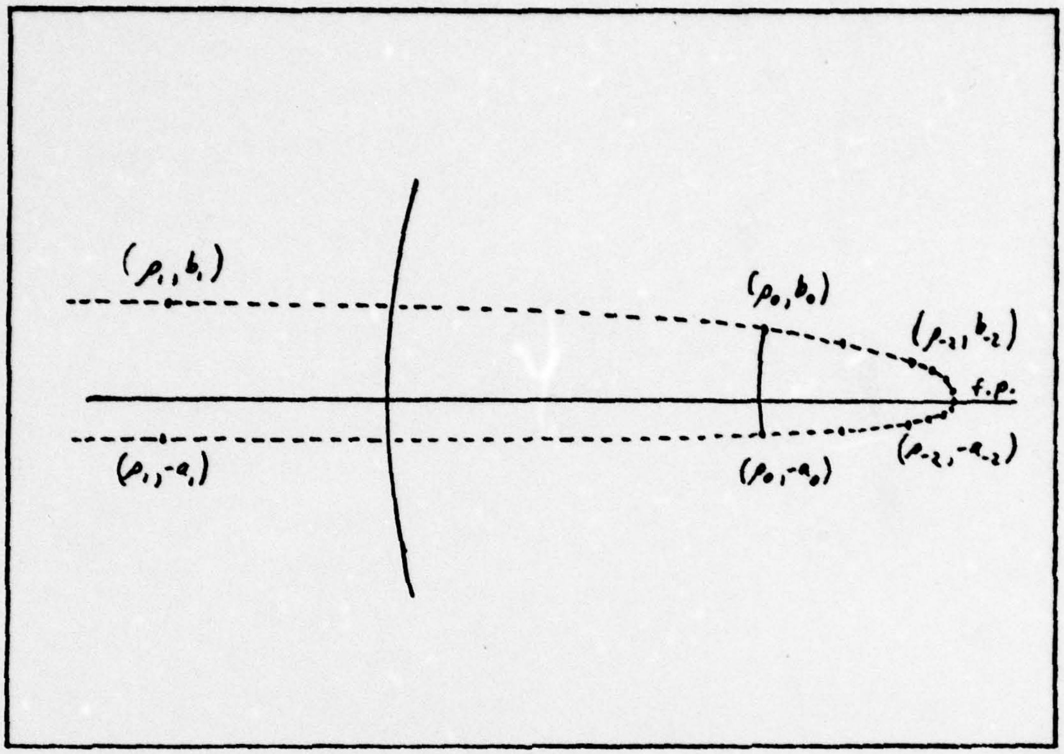

Fig. 8 VIRTUAL IMAGE LOCATIONS OF MIRROR EDGES

Appendix C

Asymptotic Expansion of Integrals by Method of Stationary Phase

This appendix outlines the development of the asymptotic expansion about the stationary phase point of integrals with rapidly oscillating exponentials. This development generally follows that of Erdeli (Ref 1:51-56). The general form of the expansion is then applied to the integral form whose specific asymptotic expansion is used in Chapter III.

General Expansion

The general form of the integral to be expanded is

$$I(x) = \int_{\alpha}^{\beta} g(t)e^{ixh(t)}dt \tag{C-1}$$

where x is a large real variable and $h(t)$ is a real function of a real variable. The stationary phase point is defined as the point $t=t_0$ where $h'(t)=0$ is satisfied. The major contribution to the integral occurs about this point. For other values of t the exponential is rapidly oscillating and the average contribution is very small. The other major contributors to the integral are the endpoints. The integral of Eq.(C-1) is broken up into two integrals at the stationary phase point t_0 . This is done to place the stationary point

at an endpoint of the integration, this is needed for the expansion. Then Eq.(C-1) becomes

$$I(x) = \left[\int_{\alpha}^{t_0} + \int_{t_0}^{\beta}\right] g(t)e^{ixh(t)}dt \qquad \text{(C-2)}$$

Each of these integrals must meet four requirements before being expanded. First, t_0 is the only stationary point on the integration interval. Second, $h(t)$ is strictly increasing on the integration interval. Third, the endpoints of the integral are stationary point of order $\rho-1$ for the lower endpoint and $\sigma-1$ for the upper endpoint where ρ and σ are greater than or equal to one. These orders can be found by expressing the derivative of $h(t)$ in the form

$$h'(t) = (t-\alpha)^{\rho-1}(\beta-t)^{\sigma-1}H(t) \qquad \text{(C-3)}$$

where $H(t)$ is any function that permits $h'(t)$ to be written in this form. The fourth requirement is that $g(t)$ not be zero at $t=t_0$.

From the third requirement, integrals of the form of Eq. (C-1) fall into three catagories:

1. For a stationary point at the upper limit of integration, $\rho=1$ and $\sigma>1$.
2. For a stationary point at the lower limit of integration, $\rho>1$ and $\sigma=1$.

3. For no stationary points on the interval of integration, $\rho=1$ and $\sigma=1$.

An integral of the Eq.(C-1) form can be expressed

$$\int_{\alpha}^{\beta} g(t)e^{ixh(t)}dt = B(x)-A(x) \tag{C-4}$$

where

$$B(x) = \int_{\alpha+\eta}^{\beta} \left[1-\nu(t)\right] g(t)e^{ixh(t)}dt \tag{C-5}$$

$$-A(x) = \int_{\alpha}^{\beta-\eta} \nu(t)g(t)e^{ixh(t)}dt \tag{C-6}$$

and

$$\nu(t) = \begin{cases} 1 \;;\alpha\leq t\leq\alpha+\eta \\ \\ 0 \;;\beta-\eta\leq t\leq\beta \end{cases} \tag{C-7}$$

The $\nu(t)$ function is a Van der Corput neutralizer, an N-times differentiable function on the interval of integration. It is used to isolate the contribution to the integral from an endpoint. This neutralizer is illustrated in Fig.(9).

The $B(x)$ term is evaluated using integration by parts. A change of variable is made where the new variable of integration, ω , is given by

$$h(t) = h(B)-\omega^{\sigma} \tag{C-8}$$

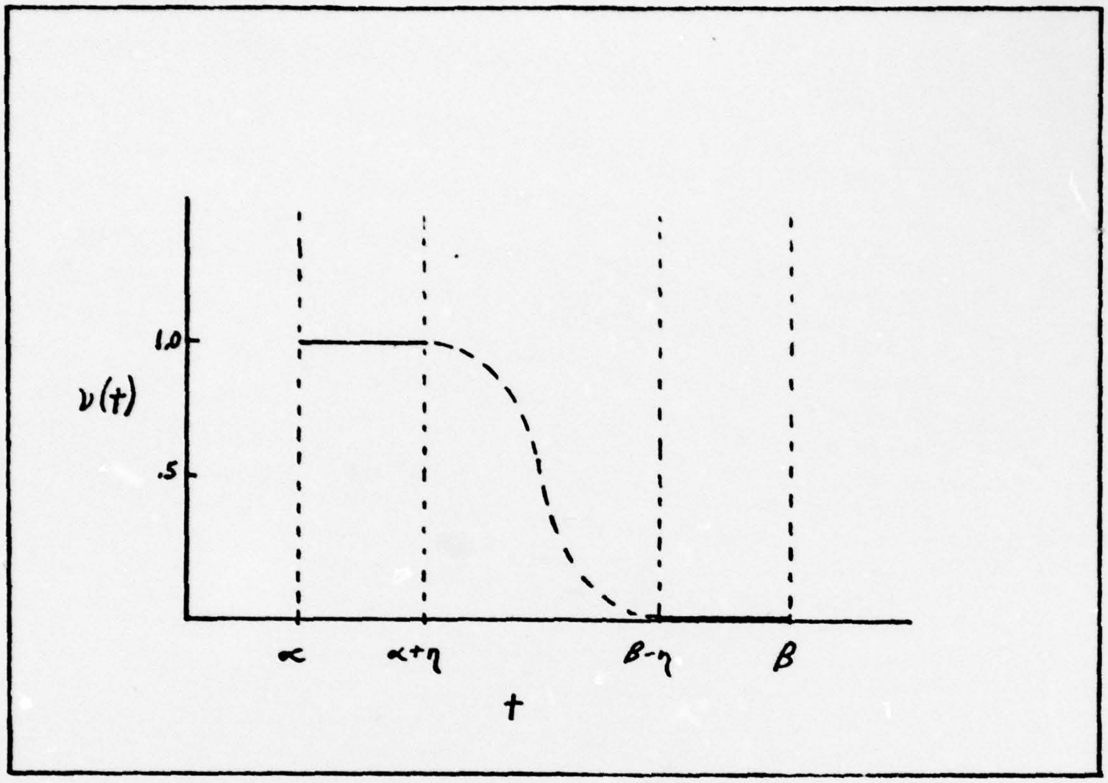

Fig. 9 BEHAVIOR OF VAN DER CORPUT NEUTRALIZER

From this the new limits of integration become, for $t=\beta$

$$\omega_1 = 0 \tag{C-9}$$

and for $t=(\alpha+\eta)$

$$\omega_2 = \left[h(\beta)-h\ (\alpha+\eta)\right]^{\frac{1}{2}} \tag{C-10}$$

A new neutralizer can be written

$$\delta(\omega) = 1-\nu(t) \tag{C-11}$$

Its values at the endpoints are

$$\delta(\omega) = \begin{cases} 1 & ,\omega=\omega_1 \\ 0 & ,\omega=\omega_2 \end{cases} \tag{C-12}$$

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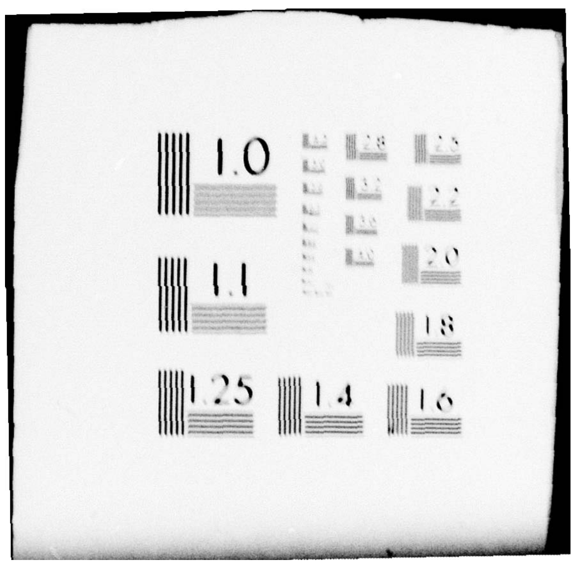
1.0
2.8
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

Also the function $y(\omega)$ is defined

$$y(\omega) = g(t)\,\frac{dt}{d\omega} \tag{C-13}$$

Using Eq.(C-8) through Eq.(C-13) the B(x) integral becomes

$$B(x) = -e^{ikh(\beta)} \int_0^{\omega_2} y(\omega)\ \delta(\omega) e^{-ix\omega^{\sigma}} d\omega \tag{C-14}$$

To integrate by parts the following are chosen

$$u = y(\omega)\,\delta(\omega)\ ;\ du = \frac{d}{d\omega}\left[y(\omega)\,\delta(\omega)\right] \tag{C-15}$$

$$dv = e^{-ix\omega^{\sigma}}\ ;\ v = E_{\sigma}(-x,\omega) \tag{C-16}$$

where $E\ (-x,\omega)$ is some function satisfying Eq.(C-16).

The integration by parts of Eq.(C-14) yields

$$\begin{aligned} B(x) &= e^{ixh(\beta)} y(\omega)\,\delta(\omega) E_{\sigma}(-x,\omega) e^{ix\omega^{\sigma}}\Big|_0^{\omega_2} \\ &\quad + e^{ixh(\beta)} \int_0^{\omega_2} E_{\sigma}(-x,\omega)\,\frac{d}{d\omega}\left[y(\omega)\,\delta(\omega)\right] d\omega \end{aligned} \tag{C-17}$$

Here it should be noted that to carry the asymptotic expansion to higher order terms the integral on the right side of Eq.(C-17) is integrated by parts. For this analysis, terms from this second integration are neglected.

Using Eq.(C-12) the neutralizer isolated the contribution to the B(x) to the $\omega_1=0$ endpoint (from $t=\beta$):

$$B(x) = e^{ixh(\beta)}y(0)E_\sigma(-x,0) \tag{C-18}$$

A function $E_\sigma(-x,\omega)$ is needed which satisfied

$$\frac{d}{d\omega} E_\sigma(-x,\omega) = e^{-ix\omega^\sigma} \tag{C-19}$$

A function satisfying this is given by the contour integral

$$\phi_{-n-1}(\omega) = \frac{(-1)^{n+1}}{n!}\int_\omega^{\infty\gamma} (z-\omega)^n e^{ixz^\sigma} dz \tag{C-20}$$

where

$$\gamma = \exp\left| i \arg(z-\omega)\right| = \exp\left| - \frac{i\pi}{2\sigma}\right| \tag{C-21}$$

Then for the $n=0$ case

$$\frac{d}{d\omega} \phi_{-1}(\omega) = \frac{d}{d\omega}\int_\omega^{\infty\gamma} (z-\omega)^0 e^{-ixz^\sigma} dz \tag{C-22}$$

$$\frac{d}{d\omega} \phi_{-1}(\omega) = -e^{-ix\omega^\sigma} \tag{C-23}$$

The $E_\sigma(-x,\omega)$ function is then

$$E_\sigma(-x,\omega) = -\phi_{-1}(\omega) \tag{C-24}$$

To get an expression for $E_\sigma(-x,0)$ in Eq.(C-18) the contour integration of $\theta_{-n-1}(\omega)$ must be performed along the path from ω to infinity at angle γ_ω to the real axis which is illustrated in Fig.(10).

A change of variable is made, letting

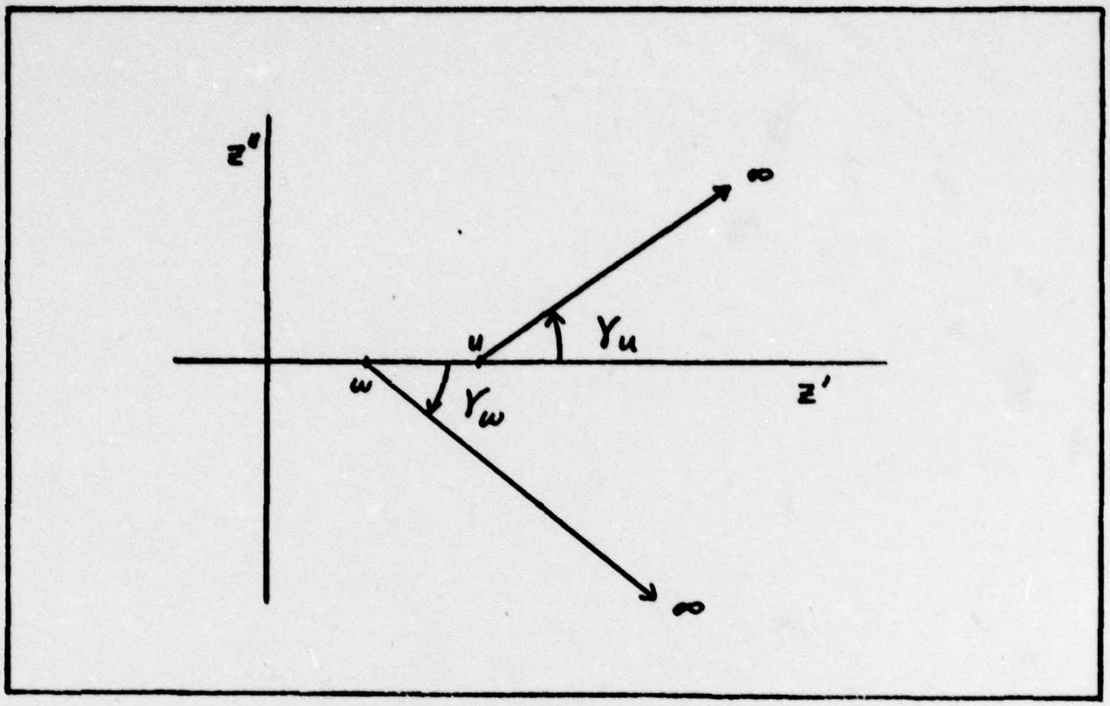

Fig. 10 PATHS OF CONTOUR INTEGRATION

$$z = te^{\frac{-i\pi}{2\sigma}} \qquad \text{(C-25)}$$

With this change of variable and letting $\omega=0$, Eq.(C-20) becomes

$$\phi_{-n-1}(0) = -e^{\frac{-i\pi}{2\sigma}(n+1)}\int_0^\infty t^n e^{xt^\sigma} dt \qquad \text{(C-26)}$$

Letting $s=-xt^\sigma$, this equation is then

$$\phi_{-n-1}(0) = -\sigma x^{\frac{-(n+1)}{\sigma}} e^{\frac{-i\pi}{2\sigma}(n+1)}$$

$$\times \int_0^\infty s^{\left(\frac{n+1}{\sigma} -1\right)} e^{-s} ds \qquad \text{(C-27)}$$

The integral term in this equation is a gamma function as defined by

$$\Gamma(m) \equiv \int_0^\infty s^{m-1} e^{-s} ds \tag{C-28}$$

where $m = \frac{n+1}{\sigma}$. Then for $n = 0$ and using Eq.(C-24)

$$E_\sigma(-x,0) = \sigma^{-1} x^{-\frac{1}{\sigma}} e^{-\frac{i\pi}{2\sigma}} \Gamma\left(\frac{1}{\sigma}\right) \tag{C-29}$$

From this, for $\sigma=1$

$$E_1(-x,0) = x^{-1} e^{-\frac{i\pi}{2}} \Gamma(1) = \frac{1}{ix} \tag{C-30}$$

And for $\sigma=2$

$$E_2(-x,0) = -\tfrac{1}{2} x^{-\frac{1}{2}} e^{-\frac{i\pi}{4}} \Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\frac{\pi}{ix}} \tag{C-31}$$

Similarly, the $-A(x)$ term of Eq.(C-6) can be found to be

$$-A(x) = -e^{ixh(\alpha)} k(0) E_\rho(x,0) \tag{C-32}$$

where Eq.(C-6) has been integrated by parts using

$$h(t) = h(\alpha) + u^\rho \tag{C-33}$$

$$k(u) = g(t) \frac{dt}{du} \tag{C-34}$$

and also using the following to get the final form

$$E_\rho(x,u) = \phi_{-1}(u) \qquad \text{(C-35)}$$

$$E_\rho(x,0) = -\rho^{-1} x^{-\frac{1}{\rho}} e^{\frac{i\pi}{2\rho}} \Gamma\left(\frac{1}{\rho}\right) \qquad \text{(C-36)}$$

Then for $\rho=1,2$

$$E_1(x,0) = \frac{1}{ix} \qquad \text{(C-37)}$$

$$E_2(x,0) = -\tfrac{1}{2}\sqrt{\frac{\pi}{ix}} \qquad \text{(C-38)}$$

Finally, Eq.(C-4) yields the general form of the asymptotic expansion about the stationary point, t_0, using Eq. (C-18) and Eq.(C-32):

$$\int_\alpha^\beta g(t)e^{ixh(t)}dt = y(0)E_\sigma(-x,0)e^{ixh(\beta)} - k(0)E_\rho(x,0)e^{ixh(\alpha)} \qquad \text{(C-39)}$$

An Application of the Asymptotic Expansion

Given an integral of the form of Eq.(C-1) that is to be approximated by an asymptotic expansion about its stationary phase point, five steps are used:

1. The stationary point of $h(t)$ is found such that $h'(t_0)=0$
2. The integral is broken up into two integrals at its

stationary point, t_0, as Eq.(C-2).

3. Any changes of variable are made that are necessary to ensure $h(t)$ in each of the two integrals is strictly increasing over the interval of integration.
4. The value of ρ and σ for each integral is determined by expressing $h'(t)$ in the form of Eq.(C-3).
5. Each of the integrals is solved as in Eq.(C-39).

For this example, the integral to be solved is

$$I(x) = \int_{\alpha}^{\beta} g(t) \exp \left| -ix(at^2-2bt+c) \right| dt \qquad \text{(C-40)}$$

The stationary point is determined from

$$h'(t) = 2(at-b) \qquad \text{(C-41)}$$

This yields the stationary point, $t_0 = \frac{b}{a}$, for $h'(t)=0$. It is assumed that $\alpha < \frac{b}{a} < \beta$.

Then Eq.(C-40) is broken up into two integrals about the stationary point $\frac{b}{a}$

$$I(x) = \left[\int_{\alpha}^{b/a} + \int_{b/a}^{\beta} \right] g(t) \exp \left| -ix(at^2-2bt+c) \right| dt \qquad \text{(C-42)}$$

For the second integral $h(t)$ is strictly decreasing on the interval of integration. A change of variables, $t=-t$, is performed to make this strictly increasing. Then Eq.(42) can be expressed as $I=I_1+I_2$, where

$$I_1 = \int_{\alpha}^{b/a} g(t) \exp\left| -ix(at^2-2bt+c)\right| dt \qquad \text{(C-43)}$$

$$I_2 = \int_{-\beta}^{-b/a} g(-t) \exp\left| -ix(at^2+2bt+c)\right| dt \qquad \text{(C-44)}$$

The first of the two integrals, I_1 , is now expanded. In the form of Eq.(C-1), $h(t)$ is identified as

$$h(t) = -(at^2-2bt+c) \qquad \text{(C-45)}$$

Expressing $h'(t)$ in the form given in Eq.(C-3) yields

$$h'(t) = 2a\left(\frac{b}{a} - t\right) \qquad \text{(C-46)}$$

From this it can be seen that $\rho=1$, $\sigma=2$, and $H(t)=2a$. Then using Eq.(C-39)

$$\begin{aligned} I_1 = {} & y(0)E_2(-x,0)e^{ixh\left(\frac{b}{a}\right)} \\ & -k(0)E_1(x,0)e^{ixh(\alpha)} \end{aligned} \qquad \text{(C-47)}$$

From Eq.(C-45)

$$h\left(\frac{b}{a}\right) = \frac{b^2}{a} - c \qquad \text{(C-48)}$$

$$h(\alpha) = -(a\alpha^2-2b\alpha+c) \qquad \text{(C-49)}$$

Next, the $y(0)$ term is determined. From Eq.(C-8) it is found that

$$\omega^2 = at - 2bt + \frac{b^2}{a} \tag{C-50}$$

Then, using the quadratic formula

$$t = \frac{b}{a} \pm \frac{\omega}{\sqrt{a}} \tag{C-51}$$

The root is chosen so that $\frac{dt}{d\omega}$ causes the resultant $y(0)$ term to be positive. In this case the positive root is chosen and so

$$\frac{dt}{d\omega} = \frac{1}{\sqrt{a}} \tag{C-52}$$

Using Eq.(C-13), Eq.(C-51) and this expression, the $y(\omega)$ term at $\omega=0$ is

$$y(0) = g\left(\frac{b}{a}\right) \frac{1}{\sqrt{a}} \tag{C-53}$$

Next the $k(0)$ term is evaluated. From Eq.(C-33) (Recalling that $\rho=1$ for this case), Eq.(C-45), and Eq.(C-49)

$$u = -at^2 + 2bt + (a\alpha^2 - 2b\alpha) \tag{C-54}$$

Using the quadratic formula

$$t = \frac{b}{a} \mp \left(\alpha^2 - 2\frac{b}{a}\alpha + \frac{b^2}{a^2} - \frac{u}{a}\right)^{\frac{1}{2}} \tag{C-55}$$

Here the root is chosen so that $\frac{dt}{du}$ will cause $k(0)$ to be negative. For this case the positive root is chosen, then

$$\frac{dt}{du} = -\tfrac{1}{2}(a^2\alpha^2-2ba\alpha+b^2-au)^{-\frac{1}{2}} \tag{C-56}$$

Using Eq.(C-34),Eq.(C-55), and this expression, the $k(u)$ for $u=0$ becomes

$$k(0) = \tfrac{1}{2}g(\alpha)\left[a\alpha-b\right]^{-1} \tag{C-57}$$

Now I_1 from Eq.(C-47) can be expressed, using Eq.(C-31), Eq.(C-37), Eq.(C-48), Eq.(C-49), Eq.(C-53), and Eq.(C-57)

$$\begin{aligned} I_1 = {} & \tfrac{1}{2}g\left(\frac{b}{a}\right)\sqrt{\frac{-i\pi}{xa}}\ \exp\left|-ix\left(c-\frac{b^2}{a}\right)\right| \\ & +g(\alpha)\left[2ix(a\alpha-b)\right]^{-1}\exp\left|-ix(a\alpha^2-2b\alpha+c)\right| \end{aligned} \tag{C-58}$$

By a similar procedure I_2 is found to be

$$\begin{aligned} I_2 = {} & \tfrac{1}{2}g\left(\frac{b}{a}\right)\sqrt{\frac{-i\pi}{xa}}\ \exp\left|-ix\left(c-\frac{b^2}{a}\right)\right| \\ & +g(\beta)\left[-2ix(a\beta-b)\right]^{-1} \\ & \times \exp\left|-ix(a\beta^2-2b\beta+c)\right| \end{aligned} \tag{C-59}$$

Finally, the integral from Eq.(C-40) is expressed in an asymptotic expansion to terms of order x^{-1} as

$$\begin{aligned} & \int_\alpha^\beta g(t)\exp\left|-ix(at^2-2bt+c)\right|\,dt \\ & = \left(\frac{-i\pi}{xa}\right)^{\frac{1}{2}} g\left(\frac{b}{a}\right)\exp\left|-ix\left(c-\frac{b^2}{a}\right)\right| \end{aligned}$$

$$+g(\alpha)\left[2ix(a\alpha-b)\right]^{-1}\exp\left\{-ix(a\alpha^2-2b\alpha+c)\right\} \tag{C-60}$$

$$+g(\beta)\left[-2ix(a\beta-b)\right]^{-1}\exp\left\{-ix(a\beta^2-2b\beta+c)\right\}$$

Appendix D

Asymptotic Approximation of Diffraction Integral for Off-Center Unstable Resonator

This appendix develops the asymptotic expansion for the diffraction integral given in Eq.(39) where $f(D,a\eta)$ is replaced by the form in Eq.(26). The form of Eq.(39) is then

$$g(\rho,\theta) = CI = C(I_1+I_2+I_3) \tag{D-1}$$

where

$$C = \frac{1}{2\pi}\left[\frac{2\pi i k a^2}{d\left(\frac{1}{\rho} - \frac{1}{d}\right)}\right]^{\frac{1}{2}} e^{2ikD} \tag{D-2}$$

$$I_1 = \sum_{n=1}^{N} (\rho_n - d)^{-\frac{1}{2}} \int_{-1}^{b/a} \exp\left\{\frac{ik}{2}\,\frac{(a\eta - b_n)^2}{\rho_n - d}\right\}$$

$$\times\ r_n(d,a\eta) \exp\left\{\frac{-ik}{2}\,\frac{\left(\theta - \frac{a}{d}\eta\right)^2}{\left(\frac{1}{\rho} - \frac{1}{d}\right)}\right\} d\eta \tag{D-3}$$

$$I_2 = \sum_{n=1}^{N} (\rho_n - d)^{-\frac{1}{2}} \int_{-1}^{b/a} \exp\left\{\frac{ik}{2}\,\frac{(a\eta + a_n)^2}{\rho_n - d}\right\}$$

$$\times\ s_n(d,a\eta) \exp\left\{\frac{-ik}{2}\,\frac{\left(\theta - \frac{a}{d}\eta\right)^2}{\left(\frac{1}{\rho} - \frac{1}{d}\right)}\right\} d\eta \tag{D-4}$$

$$+F(\eta_2)\left[2i(A\eta_2-B)\right]^{-1}\exp\left| i(A\eta_2{}^2-2B\eta_2+C)\right|$$

Note that this agrees with the asymptotic expansion form Eq.(C-60) developed in the previous appendix by another method.

Approximation for I_1

When I_1 is put into the form Eq.(D-7) the following terms are found

$$\eta_1 = -1\ ,\ \eta_2 = \frac{b}{a} \tag{D-9}$$

$$F(\eta) = r_n(d,a\eta) \tag{D-10}$$

$$A_1 = 2\pi F_{ea}\ \frac{d(\rho-d)+\rho(\rho_n-d)}{(\rho-d)(\rho_n-d)} \tag{D-11}$$

$$B_1 = 2\pi F_{ea}\ \frac{\rho\theta d(\rho_n-d)+db_n(\rho-d)}{a(\rho-d)(\rho_n-d)} \tag{D-12}$$

$$C_1 = 2\pi F_{ea}\ \frac{\rho\theta^2 d^2(\rho_n-d)+db_n{}^2(\rho-d)}{a^2(\rho-d)(\rho_n-d)} \tag{D-13}$$

Using Eq.(D-11) through Eq.(D-13) it can be shown that

$$\frac{B_1}{A_1} = \frac{d}{a}\ \frac{b_n(\rho-d)+\rho\theta(\rho_n-d)}{d(\rho-d)+\rho(\rho_n-d)} \tag{D-14}$$

$$I_3 = \int_{-1}^{b/a} \hat{f}(d,a\eta)\ \exp\left\{\frac{-ik}{2}\ \frac{\left(\theta - \frac{a}{d}\ \eta\right)^2}{\left(\frac{1}{\rho} - \frac{1}{d}\right)}\right\} d_\eta \qquad \text{(D-5)}$$

An asymptotic series approximation of integrals I_1, I_2, and I_3 can be made when they are in the form (Ref 3: 1541)

$$I = \int_{\eta_1}^{\eta_2} F(\eta)\ \exp\left|-it(A'\eta^2 - 2B'\eta + C')\right|\ d\eta \qquad \text{(D-6)}$$

where t is large valued. In I_1, I_2, and I_3 let t correspond to $-2\pi F_{ea}$ where F_{ea} is large valued and given by Eq.(3).

The Eq.(D-6) can be written

$$I = \int_{\eta_1}^{\eta_2} F(\eta)\ \exp\left|i(A\eta^2 - 2B\eta + C)\right|\ d\eta \qquad \text{(D-7)}$$

where

$$A = -2\pi F_{ea}A'$$

$$B = -2\pi F_{ea}B'$$

$$C = -2\pi F_{ea}C'$$

The asymptotic expansion of this integral when terms of higher order than F_{ea}^{-1} are neglected is

$$I = \left[\frac{i\pi}{A}\right]^{\frac{1}{2}} F\left(\frac{B}{A}\right)\ \exp\left|\ i\left(C - \frac{B^2}{A}\right)\right| \qquad \text{(D-8)}$$

$$+F(\eta_1)\ \left[-2i(A\eta_1 - B)\right]^{-1} \exp\left|\ i(A\eta_1{}^2 - 2B\eta_1 + C)\right|$$

$$I_3 = \int_{-1}^{b/a} \hat{f}(d,a\eta)\ \exp\left\{\frac{-ik}{2}\,\frac{\left(\theta-\frac{a}{d}\,\eta\right)^2}{\left(\frac{1}{\rho}-\frac{1}{d}\right)}\right\} d_\eta \qquad \text{(D-5)}$$

An asymptotic series approximation of integrals I_1 , I_2 , and I_3 can be made when they are in the form (Ref 3: 1541)

$$I = \int_{\eta_1}^{\eta_2} F(\eta)\ \exp\left|-it(A'\eta^2-2B'\eta+C')\right|\ d\eta \qquad \text{(D-6)}$$

where t is large valued. In I_1 , I_2 , and I_3 let t correspond to $-2\pi F_{ea}$ where F_{ea} is large valued and given by Eq.(3).

The Eq.(D-6) can be written

$$I = \int_{\eta_1}^{\eta_2} F(\eta)\ \exp\left|i(A\eta^2-2B\eta+C)\right|\ d\eta \qquad \text{(D-7)}$$

where

$$A = -2\pi F_{ea}A'$$
$$B = -2\pi F_{ea}B'$$
$$C = -2\pi F_{ea}C'$$

The asymptotic expansion of this integral when terms of higher order than F_{ea}^{-1} are neglected is

$$I = \left[\frac{i\pi}{A}\right]^{\frac{1}{2}} F\left(\frac{B}{A}\right)\ \exp\left|\ i\left(C-\frac{B^2}{A}\right)\right| \qquad \text{(D-8)}$$

$$+F(\eta_1)\ \left[-2i(A\eta_1-B)\right]^{-1} \exp\left|\ i(A\eta_1{}^2-2B\eta_1+C)\right|$$

$$+F(\eta_2)\left[2i(A\eta_2-B)\right]^{-1}\exp\left\{i(A\eta_2{}^2-2B\eta_2+C)\right\}$$

Note that this agrees with the asymptotic expansion form Eq.(C-60) developed in the previous appendix by another method.

Approximation for I_1

When I_1 is put into the form Eq.(D-7) the following terms are found

$$\eta_1 = -1\ ,\ \eta_2 = \frac{b}{a} \tag{D-9}$$

$$F(\eta) = r_n(d,a\eta) \tag{D-10}$$

$$A_1 = 2\pi F_{ea}\,\frac{d(\rho-d)+\rho(\rho_n-d)}{(\rho-d)(\rho_n-d)} \tag{D-11}$$

$$B_1 = 2\pi F_{ea}\,\frac{\rho\theta d(\rho_n-d)+db_n(\rho-d)}{a(\rho-d)(\rho_n-d)} \tag{D-12}$$

$$C_1 = 2\pi F_{ea}\,\frac{\rho\theta^2 d^2(\rho_n-d)+db_n{}^2(\rho-d)}{a^2(\rho-d)(\rho_n-d)} \tag{D-13}$$

Using Eq.(D-11) through Eq.(D-13) it can be shown that

$$\frac{B_1}{A_1} = \frac{d}{a}\,\frac{b_n(\rho-d)+\rho\theta(\rho_n-d)}{d(\rho-d)+\rho(\rho_n-d)} \tag{D-14}$$

$$C_1 - \frac{B_1{}^2}{A_1} = \frac{k}{2}\ \frac{\rho(b_n - \theta d)^2}{d(\rho - d) + \rho(\rho_n - d)} \tag{D-15}$$

$$A_1\eta_1 - B_1 = -\frac{ka}{2}\left[\frac{a+b_n}{\rho_n - d} + \frac{\rho(a+\theta d)}{d(\rho - d)}\right] \tag{D-16}$$

$$A_1\eta_2 - B_1 = \frac{ka}{2}\left[\frac{b-b_n}{\rho_n - d} + \frac{\rho(b-\theta d)}{d(\rho - d)}\right] \tag{D-17}$$

$$A_1\eta_1{}^2 - 2B_1\eta_1 + C_1 = \frac{k}{2}\left[\frac{(a+b_n)^2}{\rho_n - d} + \frac{\rho(a+\theta d)^2}{d(\rho - d)}\right] \tag{D-18}$$

$$A_1\eta_2{}^2 - 2B_1\eta_2 + C_1 = \frac{k}{2}\left[\frac{(b-b_n)^2}{\rho_n - d} + \frac{\rho(b-\theta d)^2}{d(\rho - d)}\right] \tag{D-19}$$

Substituting these terms into Eq.(D-8) yields

$$I_1 = \sum_{n=1}^{N}(\rho_n - d)^{-\frac{1}{2}}\left\{\left[\frac{2\pi i}{ka^2}\ \frac{d(\rho-d)(\rho_n-d)}{d(\rho-d)+\rho(\rho_n-d)}\right]^{\frac{1}{2}}\right.$$

$$\times\ r_n\left(d, d\,\frac{\rho\theta(\rho_n-d)+b_n(\rho-d)}{d(\rho-d)+\rho(\rho_n-d)}\right)\exp\left|\ \frac{ik}{2}\ \frac{\rho(b_n-\theta d)^2}{d(\rho-d)+\rho(\rho_n-d)}\right|$$

$$+\ r_n(d,-a)\left[ika\left(\frac{a+b_n}{\rho_n-d} + \frac{\rho(a+\theta d)}{d(\rho-d)}\right)\right]^{-1} \tag{D-20}$$

$$\times\ \exp\left|\frac{ik}{2}\left(\frac{(a+b_n)^2}{\rho_n-d} + \frac{\rho(a+\theta d)^2}{d(\rho-d)}\right)\right|$$

$$+r(d,b)\left[ika\left(\frac{b-b_n}{\rho_n-d} + \frac{\rho(b-\theta d)}{d(\rho-d)}\right)\right]^{-1}$$

$$\times \exp\left|\frac{ik}{2}\left(\frac{(b-b_n)^2}{\rho_n-d}+\frac{\rho(b-\theta d)^2}{d(\rho-d)}\right)\right|\Bigg\}$$

<u>Approximation for I_2</u>

The limits of integration are the same as Eq.(D-9). When I_2 is put into the form of Eq.(D-7) the following terms are found

$$F(\eta) = s_n(d,a\eta) \tag{D-21}$$

$$A_2 = 2\pi F_{ea}\frac{d(\rho-d)+\rho(\rho_n-d)}{(\rho-d)\ (\rho_n-d)} \tag{D-22}$$

$$B_2 = 2\pi F_{ea}\frac{\rho\theta d(\rho_n-d)-a_n(\rho-d)}{a(\rho-d)\ (\rho_n-d)} \tag{D-23}$$

$$C_2 = 2\pi F_{ea}\frac{\rho\theta^2 d^2(\rho_n-d)+da_n^2(\rho-d)}{a^2(\rho-d)(\rho_n-d)} \tag{D-24}$$

It then follows that

$$\frac{B_2}{A_2} = \frac{d}{a}\ \frac{\rho\theta(\rho_n-d)-a_n(\rho-d)}{d(\rho-d)\ +\rho(\rho_n-d)} \tag{D-25}$$

$$C_2-\frac{B_2^{\,2}}{A_2} = \frac{k}{2}\ \frac{\rho(a_n+\theta d)^2}{d(\rho-d)+\rho(\rho_n-d)} \tag{D-26}$$

$$A_2\eta_1-B_2 = -\frac{ka}{2}\left[\frac{a-a_n}{\rho_n-d}+\frac{\rho(a+\theta d)}{\rho-d}\right] \tag{D-27}$$

$$A_2\eta_2-B_2 = \frac{ka}{2}\left[\frac{b+a_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right] \tag{D-28}$$

$$A_2\eta_1{}^2-2B_2\eta_1+C_2 = \frac{k}{2}\left[\frac{(a-a_n)^2}{(\rho_n-d)} + \frac{\rho(a+\theta d)}{d(\rho-d)}\right] \quad \text{(D-29)}$$

$$A_2\eta_2{}^2-2B_2\eta_2+C_2 = \frac{k}{2}\left[\frac{(b+a_n)^2}{(\rho_n-d)} + \frac{\rho(b-\theta d)}{d(\rho-d)}\right] \quad \text{(D-30)}$$

Substituting into Eq.(D-8) yields

$$I_2 = \sum_{n=1}^{N}(\rho_n-d)^{-\frac{1}{2}}\left\{\left[\frac{2\pi i}{ka^2}\,\frac{d(\rho-d)(\rho_n-d)}{d(\rho-d)+\rho(\rho_n-d)}\right]^{\frac{1}{2}}\right.$$

$$X\ s_n\left(d, d\,\frac{\rho\theta(\rho_n-d)-a_n(\rho-d)}{\rho(\rho_n-d)+d(\rho-d)}\right)$$

$$X\ \exp\left\{\frac{ik}{2}\,\frac{\rho(a_n+\theta d)^2}{d(\rho-d)+\rho(\rho_n-d)}\right\}$$

$$+\ s_n(d,-a)\left[ika\left(\frac{a-a_n}{\rho_n-d} + \frac{\rho(a+\theta d)}{d(\rho-d)}\right)\right]^{-1} \quad \text{(D-31)}$$

$$X\ \exp\left\{\frac{ik}{2}\left(\frac{(a-a_n)^2}{\rho_n-d} + \frac{\rho(a+\theta d)^2}{d(\rho-d)}\right)\right\}$$

$$+\ s_n(d,b)\left[ika\left(\frac{b+a_n}{\rho_n-d} + \frac{\rho(b-\theta d)}{d(\rho-d)}\right)\right]^{-1}$$

$$\left. X\ \exp\left\{\frac{ik}{2}\left(\frac{(b+a_n)^2}{\rho_n-d} + \frac{\rho(b-\theta d)^2}{d(\rho-d)}\right)\right\}\right\}$$

<u>Approximation for I_3</u>

The limits of integration for I_3 are given in Eq.(D-9). When I_3 is put into the form of Eq.(D-7) the following terms

are found

$$F(\eta) = \hat{f}(d, a\eta) \tag{D-32}$$

$$A_3 = 2\pi F_{ea} \frac{\rho}{\rho - d} \tag{D-33}$$

$$B_3 = 2\pi F_{ea} \frac{\rho\theta d}{a(\rho - d)} \tag{D-34}$$

$$C_3 = 2\pi F_{ea} \frac{\rho\theta^2 \; d^2}{a^2(\rho - d)} \tag{D-35}$$

Then it follows that

$$\frac{B_3}{A_3} = \frac{\theta d}{a} \tag{D-36}$$

$$C_3 - \frac{B_3{}^2}{A_3} = 0 \tag{D-37}$$

$$A_3\eta_1 - B_3 = \frac{ka}{2} \; \frac{\rho(a+\theta d)}{d(\rho - d)} \tag{D-38}$$

$$A_3\eta_2 - B_3 = \frac{ka}{2} \; \frac{\rho(b-\theta d)}{d(\rho - d)} \tag{D-39}$$

$$A_3\eta_1{}^2 - 2B_3\eta_1 + C_3 = \frac{k}{2} \; \frac{\rho(a+\theta d)^2}{d(\rho - d)} \tag{D-40}$$

$$A_3\eta_2{}^2 - 2B_3\eta_2 + C_3 = \frac{k}{2} \; \frac{\rho(b-\theta d)^2}{d(\rho - d)} \tag{D-41}$$

Substituting into Eq.(D-8) yields

$$I_3 = \left[\frac{2\pi i}{ka^2} \frac{d}{\rho}(\rho - d)\right]^{\frac{1}{2}} \hat{f}(d, \theta d)$$

$$+ \hat{f}(d,-a) \left[ika \frac{\rho(a+\theta d)}{d(\rho-d)} \right]^{-1}$$

$$X \exp \left| \frac{ik}{2} \frac{\rho(a+\theta d)^2}{d(\rho-d)} \right| \tag{D-42}$$

$$+ \hat{f}(d,b) \left[ika \frac{\rho(b-\theta d)}{d(\rho-d)} \right]^{-1}$$

$$X \exp \left| \frac{ik}{2} \frac{\rho(b-\theta d)^2}{d(\rho-d)} \right|$$

When Eq.(D-2), Eq.(D-20), Eq.(D-31), and Eq.(D-42) are substituted into Eq.(D-3) the expression for $g(\rho,\theta)$ near the feedback mirror is obtained. The final result is

$$g(\rho,\theta) = \frac{1}{2\pi} \left[\frac{2\pi ika^2}{d\left(\frac{1}{\rho} - \frac{1}{d}\right)} \right]^{\frac{1}{2}} e^{2ikD}$$

$$X \left\{ \sum_{n=1}^{N} (\rho_n - d)^{-\frac{1}{2}} \left(\left[\frac{2\pi i}{ka^2} \frac{d(\rho-d)(\rho_n-d)}{d(\rho-d)+\rho(\rho_n-d)} \right]^{\frac{1}{2}} \right. \right.$$

$$X\, r_n \left(d, d \frac{\rho\theta(\rho_n-d)+b_n(\rho-d)}{\rho(\rho_n-d)+d(\rho-d)} \right) \tag{D-43}$$

$$X \exp \left| \frac{ik}{2} \frac{\rho(b_n-\theta d)^2}{\rho(\rho_n-d)+d(\rho-d)} \right|$$

$$+ r_n(d,-a) \left[ika\left(\frac{a+b_n}{\rho_n-d} + \frac{\rho(a+\theta d)}{d(\rho-d)} \right) \right]^{-1}$$

$$X \exp \left| \frac{ik}{2} \left(\frac{(a+b_n)^2}{\rho_n-d} + \frac{\rho(a+\theta d)^2}{d(\rho-d)} \right) \right|$$

$$+\ r_n(d,b)\ \left[ika\left(\frac{b-b_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right)\right]^{-1}$$

$$X\ \exp\left|\frac{ik}{2}\left(\frac{(b-b_n)^2}{\rho_n-d}+\frac{\rho(b-\theta d)^2}{d(\rho-d)}\right)\right|\Bigg)$$

$$+\sum_{n=1}^{N}(\rho_n-d)^{-\frac{1}{2}}\Bigg(\left[\frac{2\pi i}{ka^2}\ \frac{d(\rho-d)\ (\rho_n-d)}{d(\rho-d)+\rho(\rho_n-d)}\right]^{\frac{1}{2}}$$

$$X\ s_n\left(d,d\ \frac{\rho\theta(\rho_n-d)-a_n(\rho-d)}{\rho(\rho_n-d)+d(\rho-d)}\right)$$

$$X\ \exp\left|\frac{ik}{2}\ \frac{\rho(a_n+\theta d)^2}{\rho(\rho_n-d)+d(\rho-d)}\right|$$

$$+\ s_n(d,-a)\ \left[ika\left(\frac{a-a_n}{\rho_n-d}+\frac{\rho(a+\theta d)}{d(\rho-d)}\right)\right]^{-1}$$

$$X\ \exp\left|\frac{ik}{2}\left(\frac{(a-a_n)^2}{\rho_n-d}+\frac{\rho(a+\theta d)^2}{d(\rho-d)}\right)\right| \qquad \text{(D-43)}$$

$$+\ s_n(d,b)\ \left[ika\ \left(\frac{b+a_n}{\rho_n-d}+\frac{\rho(b-\theta d)}{d(\rho-d)}\right)\right]^{-1}$$

$$X\ \exp\left|\frac{ik}{2}\left(\frac{(b+a_n)^2}{\rho_n-d}+\frac{\rho(b-\theta d)^2}{d(\rho-d)}\right)\right|\Bigg)$$

$$+\Bigg(\left[\frac{2\pi i}{ka^2}\ \frac{d}{\rho}\ (\rho-d)\right]^{\frac{1}{2}}\ \hat{f}(d,\theta d)$$

$$+\ \hat{f}(d,-a)\ \left[ika\ \frac{\rho(a+\theta d)}{d(\rho-d)}\right]^{-1}$$

$$\times \exp\left| \frac{ik}{2} \frac{\rho(a+\theta d)^2}{d(\rho-d)} \right|$$

$$+ \hat{f}(d,b) \left[ika \frac{\rho(b-\theta d)}{d(\rho-d)} \right]^{-1} \tag{D-43}$$

$$\left. \left. \times \exp\left| \frac{ik}{2} \frac{\rho(b-\theta d)^2}{d(\rho-d)} \right| \right) \right\}$$

Appendix E

Elements of Coefficient Matrix $\underline{A}(\mu)$

This appendix illustrates several of the elements of the coefficient matrix $\underline{A}(\mu)$ of Eq.(140). The coefficients are of a very complex form but their values could be calculated on a computer since the eigenvalues, μ , are the only unknown parameters.

From Eq.(141), the $\underline{A}(\mu)$ matrix is square and of dimension (2N+1) . It can be expressed as

$$\underline{A} = \begin{bmatrix} M_{1,1} & M_{1,2} \cdots\cdots & M_{1,N+1} \cdots\cdots & M_{1,2N} & M_{1,2N+1} \\ & & \vdots & \vdots & \vdots \\ M_{2,1} & M_{2,2} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{2N+1,1} & M_{2N+1,2} \cdots & M_{2N+1,N+1} \cdots\cdots & \cdots\cdots & M_{2N+1,2N+1} \end{bmatrix} \qquad \text{(E-1)}$$

where $M_{k,n}$ is a matrix element.

Similarly, the q vector can be expressed

$$\underline{q} = \begin{bmatrix} q(x_N) \\ q(x_{N-1}) \\ \vdots \\ q(x_1) \\ q(x_0) \\ q(x_{-1}) \\ \vdots \\ q(x_{-N+1}) \\ q(x_{-N}) \end{bmatrix} \qquad \text{(E-2)}$$

From Eq.(139) several examples of coefficient matrix elements are

$$M_{1,1} = CE_{+N}A_{+N}(x_{+N})\mu^{N}-1 \tag{E-3}$$

$$M_{1,2} = CE_{N-1}A_{N-1}(x_{+N})\mu^{N-1} \tag{E-4}$$

$$M_{1,N+1} = 2CE_{0}\frac{\mu^{N+1}}{1-\mu} \tag{E-5}$$

$$M_{1,N+2} = CE_{-1}A_{-1}(x_{+N})\mu \tag{E-6}$$

$$M_{1,2N+1} = CE_{-N}A_{-N}(x_{+N})\mu^{N} \tag{E-7}$$

$$M_{2,2} = CE_{N-1}A_{N-1}(x_{N-1})\mu^{N-1}-1 \tag{E-8}$$

$$M_{2N+1,1} = CE_{N}A_{N}(x_{-N})\mu^{N} \tag{E-9}$$

$$M_{2N+1,2N+1} = CE_{-N}A_{-N}(x_{-N})\mu^{N}-1 \tag{E-10}$$

Then, using Eq.(134) through Eq.(138), several examples of these matrix elements in more detailed form are

$$M_{1,1} = \left(-\frac{1}{2\pi}\left(\frac{i}{2F_e}\right)\right)^{\frac{1}{2}} \times \exp\left\{2\pi i F_e \frac{M^{N}-1}{M^{N}+1}\right\}(1+M^{-N}) \tag{E-11}$$

$$\times \left[1 - \frac{aM^{-N}(1-M^{-2})}{1+M^{-N}} \left(a - \frac{a}{M} \frac{M^{-1}+M^{1-N}}{1+M^{-N}} \right)^{-1} \right]^{-1}$$

$$\left. \times \mu^{N} \right)^{-1}$$

$$M_{1,N+1} = -\frac{1}{\pi} \left(\frac{i}{2F_e} \right)^{\frac{1}{2}} \exp\{2\pi i F_e\} \frac{\mu^{N+1}}{1-\mu} \tag{E-12}$$

$$M_{1,2N+1} = -\frac{1}{2\pi} \left(\frac{i}{2F_e} \right)^{\frac{1}{2}} \mu^{N}$$

$$\times \exp\left\{ 2\pi i F_e \frac{M^N+1}{M^N-1} \right\} \left(1-M^{-N} \right) \tag{E-13}$$

$$\times \left[1 + \frac{aM^{-N}(1-M^{-2})}{(1-M^{-N})} \left(a - \frac{a}{M} \frac{M^{-1}+M^{1-N}}{1+M^{-N}} \right)^{-1} \right]^{-1}$$

VITA

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A polynomial equation for the eigenvalues of the modes of off-center unstable confocal resonators is developed. A constant gain for steady state modes in a bare cavity is assumed. The field is built-up from right and left-traveling diffraction components for a number of round trips through the resonator and geometrical components from the core region. Using an asymptotic expansion of the diffraction integral, the boundary conditions are

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developed. These, with the propagation equations across the resonator, are used to relate the diffraction and geometrical components to the diffraction amplitude after one round trip in the cavity. The polynomial equation for the eigenvalues is developed from the first round trip amplitude function, after approximating a slowly varying function of the field to be constant. A method is proposed for examining the behavior of the approximated function for the centered resonator case and including it in mode calculations if necessary.